Comment tirer parti de l'embarras du choix

face à plusieurs modèles de prévision concurrents

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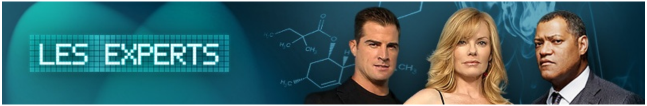
The framework of this talk

Sequential and worst-case deterministic prediction of time series based on expert advice

A statistician has to predict a time series $y_1, y_2, \ldots \in \mathcal{C}$, where $\mathcal{C}$ is a convex subset of $\mathbb{R}^d$.

Finitely many expert forecasts are available, e.g., given by some stochastic models.



At each instance t, expert $j \in \{1, \ldots, N\}$ outputs a forecast

$$f_{j,t} = f_{j,t}\left(y_1^{t-1}\right) \in \mathcal{C}$$

Observations and predictions are made in a sequential fashion:

The prediction $\widehat{y}_t$ of y_t is determined based

- on the past observations $y_1^{t-1} = (y_1, \ldots, y_{t-1})$,
- and the current and past expert forecasts $f_{j,s}$, where $s \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, N\}$,

before getting to know the actual value y_t.

A typical solution of the problem is to form convex (or linear) combinations of the expert forecasts, with weights $\mathbf{p}_t = \big(p_{1,t}, \ldots, p_{N,t}\big)$ or $\mathbf{v}_t = \big(v_{1,t}, \ldots, v_{N,t}\big)$ adjusted over time.

The statistician then outputs the forecasts $\displaystyle \widehat{y}_t = \sum_{j=1}^{N} p_{j,t}\, f_{j,t}$

The observations y_t will not be considered stochastic anymore at this stage; thus the performance criterion will be a relative one.

We consider a convex loss function $\ell : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$, e.g., the square loss $\ell(x,y) = (x-y)^2$ when $\mathcal{C} \subseteq \mathbb{R}$.

The cumulative losses of the statistician and of the constant convex combinations $\mathbf{q} = (q_1, \ldots, q_N)$ of the expert forecasts equal

$$\widehat{L}_T = \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} p_{j,t}\, f_{j,t},\ y_t\right) \quad \text{and} \quad L_T(\mathbf{q}) = \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} q_j\, f_{j,t},\ y_t\right)$$

First study: Forecasting of air quality

Starting date: September 2005

Academic partner: Vivien Mallet, INRIA, project-team CLIME

Industrial partner: Edouard Debry, INERIS (Institut National de l'EnviRonnement Industriel et des RisqueS)

M.Sc. students involved over time:

- Boris Mauricette (6 months in 2007; from M2 Pro Paris-Diderot and ENS de Lyon)
- Sébastien Gerchinovitz (5 months in 2008; from M2 Maths Paris-Sud)
- Karim Drifi (4 months in 2009; from M2 MVA ENS Cachan)
- Paul Baudin (4 months in 2012; from M2 MVA ENS Cachan)

Associated publication: in the Journal of Geophysical Research

Some characteristics of one among the studied data sets:

- 126 days during summer '01; one-day ahead prediction
- 241 stations in France and Germany
- Typical ozone concentrations between 40 μg m^{-3} and 150 μg m^{-3}; sometimes above the values 180 μg m^{-3} or 240 μg m^{-3}
- 48 experts, built in Mallet et Sportisse '06 by choosing a physical and chemical formulation, a numerical approximation scheme to solve the involved PDEs, and a set of input data (among many)

$\rightarrow$ Instead of trusting only one model/expert ("selection"), we proceed in a more greedy way and consider many models/experts, which we combine sequentially ("aggregation").

This leads to more accurate and more stable (meta-)predictions.

The stations of the network are indexed by $\mathcal{S}$.

Each model $j = 1, \ldots, 48$ outputs a prediction $f_{j,t}^s$ for the ozone peak at station s and day t, which is then compared to the measured peak y_t^s. (We discard measurement errors.)

The statistician chooses at each round a single convex weight vector $\mathbf{p}_t$ or linear weight vector $\mathbf{v}_t$ to be used at all stations; this leads to prediction fields.

The strategies are assessed based on their RMSEs, which amounts to considering the convex losses

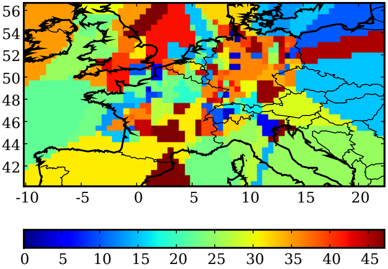
$$\ell_t(\mathbf{p}_t) \stackrel{\text{def}}{=} \sum_{s \in \mathcal{S}_t} \left(\sum_{j=1}^{48} p_{j,t} \, f_{j,t}^s - y_t^s \right)^2$$

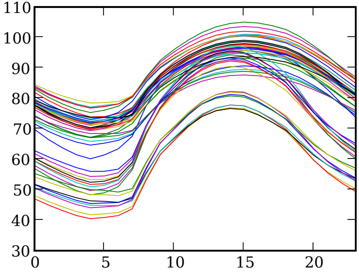
where $\mathcal{S}_t$ is the subset of active stations at day t.

The RMSE equals $\sqrt{\dfrac{\sum_{t=t_0}^{T} \ell_t(\mathbf{p}_t)}{\sum_{t=t_0}^{T} |\mathcal{S}_t|}}$ for $t_0 = 31$ (short training)

Left: There are several good and useful experts.

Right: Their forecasting profiles are quite different (the experts are not clones the ones of the others!).





Left: Coloring of Europe according to the index of the locally best expert

Right: Average forecasting profiles during a day (averages over time and space)

The framework of this talk

(continued)

The regret R_T is defined as the difference

$$\widehat{L}_T - \min_{\mathbf{q}} L_T(\mathbf{q}) = \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} p_{j,t}\, f_{j,t},\; y_t\right) - \min_{\mathbf{q}} \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} q_j\, f_{j,t},\; y_t\right)$$

We are interested in aggregation rules with (uniformly) vanishing per-round regret,

$$\limsup_{T\to\infty} \quad \frac{1}{T} \sup \left\{ \widehat{L}_T - \min_{\mathbf{q}} L_T(\mathbf{q}) \right\} \leqslant 0$$

where the supremum is over all possible sequences of observations and of expert forecasts. (Not just over most of these sequences!)

Remarks:

– Hence the name "prediction of individual sequences" (or robust aggregation of expert forecasts).

– The best convex combination $\mathbf{q}^\star$ is known in hindsight whereas the statistician has to predict in a sequential fashion.

This framework leads to a meta-statistical interpretation:

- each series of expert forecasts is given by a statistical forecasting method, possibly tuned with some given set of parameters;
- these base forecasts relying on some stochastic model are then combined in a robust and deterministic manner.

The cumulative loss of the statistician can be decomposed as

$$\widehat{L}_T = \min_{\mathbf{q}} L_T(\mathbf{q}) + R_T$$

This leads to the following interpretations:

- the term indicating the performance of the best convex combination of the expert forecasts is an approximation error;
- the regret term measures a sequential estimation error.

First study, continued

Forecasting of the air quality

How good are our experts? See the “oracles” below.

Do we expect the aggregation methods to provide significant improvements? Yes, whenever the best convex and/or linear combinations significantly outperform the best expert.

Uniform mean	Best expert	Best **p**	Best **u**
24.41	22.43	21.45	19.24

Performance, in terms of RMSE, of (some combinations of) the experts

Disclaimer

We could also consider batch learning methods to aggregate models/experts, like

- BMA (Bayesian model averaging),
- CART (classification and regression trees),
- random forests, etc.,

or even selection methods, and apply them online, by running a batch analysis at each step.

We instead resort to "real" online techniques that, in addition, come up with theoretical guarantees even in non-stochastic scenarios.

We will also see that calibrating their parameters can be done in a more satisfactory way, using the sequential character of the prediction.

A strategy to pick convex weights

Let's do some maths. But simple maths, and for 10 minutes only!

Reminder of the aim and setting:

Given a loss function $\ell : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, where $\mathcal{C} \subseteq \mathbb{R}^d$ is convex

Choose sequentially the convex weights $p_{j,t}$

To uniformly bound the regret with respect to all sequences of observations y_t and expert predictions $f_{j,t}$:

$$\sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} p_{j,t}\, f_{j,t},\; y_t\right) - \min_{\mathbf{q}} \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} q_j\, f_{j,t},\; y_t\right)$$

When ℓ is convex in its first argument, sub-gradients exist, i.e.:

For all $x, y \in \mathcal{C}$, there exists $\nabla\ell(x, y)$ such that

$$\forall x' \in \mathcal{C}, \qquad \ell(x, y) - \ell(x', y) \leqslant \nabla\ell(x, y) \cdot (x - x')$$

To uniformly bound the regret with respect to all convex weight vectors $\mathbf{q}$, we write

$$\begin{aligned}
& \max_{\mathbf{q}} \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} p_{j,t}\, f_{j,t},\, y_t\right) - \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} q_j\, f_{j,t},\, y_t\right) \\
\leqslant \quad & \max_{\mathbf{q}} \sum_{t=1}^{T} \nabla\ell\left(\sum_{k=1}^{N} p_{k,t} f_{k,t},\, y_t\right) \cdot \left(\sum_{j=1}^{N} p_{j,t} f_{j,t} - \sum_{j=1}^{N} q_j f_{j,t}\right) \\
= \quad & \max_{\mathbf{q}} \sum_{t=1}^{T} \left(\sum_{j=1}^{N} p_{j,t}\widetilde{\ell}_{j,t} - \sum_{j=1}^{N} q_j \widetilde{\ell}_{j,t}\right) \\
= \quad & \sum_{t=1}^{T} \sum_{j=1}^{N} p_{j,t}\widetilde{\ell}_{j,t} - \min_{i=1,\ldots,N} \sum_{t=1}^{T} \widetilde{\ell}_{i,t}
\end{aligned}$$

where we denoted

$$\widetilde{\ell}_{j,t} = \nabla\ell\left(\sum_{k=1}^{N} p_{k,t} f_{k,t},\, y_t\right) \cdot f_{j,t}$$

Via the (signed) pseudo-losses

$$\widetilde{\ell}_{j,t} = \nabla \ell \left(\sum_{k=1}^{N} p_{k,t} f_{k,t},\, y_t \right) \cdot f_{j,t}$$

it suffices to consider the following simplified framework.

At each round $t = 1, 2, \ldots,$

- the experts provide forecasts $f_{1,t}, \ldots, f_{N,t}$;
- the statistician picks convex weights $\mathbf{p}_t = \big(p_{1,t}, \ldots, p_{N,t}\big)$;
- the environment then determines, possibly with the knowledge of $\mathbf{p}_t$, a loss vector $\big(\widetilde{\ell}_{1,t}, \ldots, \widetilde{\ell}_{N,t}\big)$

The aim is to bound uniformly the regret

$$R_T = \sum_{t=1}^{T} \sum_{j=1}^{N} p_{j,t} \widetilde{\ell}_{j,t} - \min_{i=1,\ldots,N} \sum_{t=1}^{T} \widetilde{\ell}_{i,t}$$

For all $j \in \{1, \ldots, N\}$, we pick $p_{j,1} = 1/N$ and for all $t \geqslant 2$,

$$p_{j,t} = \frac{\exp\left(-\eta \sum_{s=1}^{t-1} \widetilde{\ell}_{j,s}\right)}{\sum_{k=1}^{N} \exp\left(-\eta \sum_{s=1}^{t-1} \widetilde{\ell}_{k,s}\right)}$$

This strategy is known as performing exponentially weighted averages of the past cumulative losses of the experts (with fixed learning rate $\eta > 0$).

Lemma. Consider two real numbers $m \leqslant M$.

For all $\eta > 0$ and for all individual sequences $\widetilde{\ell}_{j,t} \in [m, M]$,

$$R_T = \sum_{t=1}^{T} \sum_{j=1}^{N} p_{j,t} \widetilde{\ell}_{j,t} - \min_{i=1,\ldots,N} \sum_{t=1}^{T} \widetilde{\ell}_{i,t} \leqslant \frac{\ln N}{\eta} + \eta \frac{(M-m)^2}{8} T$$

References: Vovk '90; Littlestone and Warmuth '94

Proof of the regret bound

It relies on Hoeffding's lemma: for all random variables X with range $[m, M]$, for all $s \in \mathbb{R}$,

$$\ln \mathbb{E}\left[e^{sX}\right] \leqslant s\,\mathbb{E}[X] + \frac{s^2}{8}(M-m)^2$$

For all $t = 1, 2, \ldots$,

$$\begin{aligned} -\eta \sum_{j=1}^{N} p_{j,t}\widetilde{\ell}_{j,t} &= -\eta \sum_{j=1}^{N} \frac{\exp\left(-\eta \sum_{s=1}^{t-1} \widetilde{\ell}_{j,s}\right)}{\sum_{k=1}^{N} \exp\left(-\eta \sum_{s=1}^{t-1} \widetilde{\ell}_{k,s}\right)} \widetilde{\ell}_{j,t} \\ &\geqslant \ln \frac{\sum_{j=1}^{N} \exp\left(-\eta \sum_{s=1}^{t} \widetilde{\ell}_{j,s}\right)}{\sum_{k=1}^{N} \exp\left(-\eta \sum_{s=1}^{t-1} \widetilde{\ell}_{k,s}\right)} - \frac{\eta^2}{8}(M-m)^2 \end{aligned}$$

A telescoping sum appears and leads to

$$\sum_{t=1}^{T} \sum_{j=1}^{N} p_{j,t}\widetilde{\ell}_{j,t} \leqslant \underbrace{-\frac{1}{\eta} \ln \frac{\sum_{j=1}^{N} \exp\left(-\eta \sum_{s=1}^{T} \widetilde{\ell}_{j,s}\right)}{N}}_{\leqslant \min_{i=1,\ldots,N} \sum_{t=1}^{T} \widetilde{\ell}_{i,t} + \frac{\ln N}{\eta}} + \eta \frac{(M-m)^2}{8} T\,.$$

We now discuss the obtained bound.

Recall that $[m, M]$ is the loss range.

The stated bound can be optimized in η:

$$R_T \leqslant \min_{\eta>0} \left\{ \frac{\ln N}{\eta} + \eta \frac{(M-m)^2}{8} T \right\} = (M-m)\sqrt{\frac{T}{2}\ln N}$$

for the (theoretical) optimal choice

$$\eta^\star = \frac{1}{M-m}\sqrt{\frac{8\ln N}{T}}$$

This choice depends on M and m, which are sometimes not known beforehand, as well as on T, which may not be bounded (if the prediction game goes forever).

Since no fixed value of $\eta > 0$ ensures that $R_T = o(T)$, we still have no fully sequential strategy... but this can be taken care of.

The possibles patches are, first, to resort to the "doubling trick."

Alternatively, the learning rates of the exponentially weighted average strategy may vary over time, depending on the past: for $t \geqslant 2$,

$$p_{j,t} = \frac{\exp\left(-\eta_t \sum_{s=1}^{t-1} \widetilde{\ell}_{j,s}\right)}{\sum_{k=1}^{N} \exp\left(-\eta_t \sum_{s=1}^{t-1} \widetilde{\ell}_{k,s}\right)}$$

By a careful such adaptive choice of the η_t, the following regret bound can be obtained:

$$R_T \leqslant \square\,(M-m)\sqrt{T \ln N} + \square\,(M-m) \ln N$$

where the $\square$ denote some universal constants.

We thus recover the same orders of magnitude for the regret bound.

References: Auer, Cesa-Bianchi and Gentile '02; Cesa-Bianchi, Mansour and Stoltz '07

However, these theoretically satisfactory solutions would not work well in practice.

This is what we do instead. (It is very different from techniques like cross-validation: we exploit the sequential fashion.)

The exponentially weighted average strategy $\mathcal{E}_\eta$ with fixed learning rate η picks the convex combination $\mu_t(\eta)$, where

$$p_{j,t}(\eta) = \frac{\exp\left(-\eta \sum_{s=1}^{t-1} \widetilde{\ell}_{j,s}\right)}{\sum_{k=1}^{N} \exp\left(-\eta \sum_{s=1}^{t-1} \widetilde{\ell}_{k,s}\right)}$$

We denote its cumulative loss $\widehat{L}_t(\eta) = \displaystyle\sum_{s=1}^{t} \ell\left(\sum_{j=1}^{N} p_{j,s}(\eta) f_{j,s},\ y_s\right)$

Based on the family of the $\mathcal{E}_\eta$, we build a data-driven meta-strategy which at each instance $t \geqslant 2$ resorts to

$$\mathbf{p}_{t+1}(\eta_t) \qquad \text{where} \qquad \eta_t \in \mathop{\arg\min}_{\eta>0} \widehat{L}_t(\eta)$$

Reference: An idea of Vivien Mallet

Other natural variants: Focus on the most recent losses

Moving sums (with window of size H):

$$p_{j,t} = \frac{\exp\left(-\eta \sum_{s=\max\{1,t-H\}}^{t-1} \widetilde{\ell}_{j,s}\right)}{\sum_{k=1}^{N} \exp\left(-\eta \sum_{s=\max\{1,t-H\}}^{t-1} \widetilde{\ell}_{k,s}\right)}$$

One can prove that the regret is $\geqslant \square\, T$ in the worst case.

Discounted losses (with discounts given by a sequence $\beta_t \searrow 0$):

$$p_{j,t} = \frac{\exp\left(-\eta_t \sum_{s=1}^{t-1} (1+\beta_{t-s}) \widetilde{\ell}_{j,s}\right)}{\sum_{k=1}^{N} \exp\left(-\eta_t \sum_{s=1}^{t-1} (1+\beta_{t-s}) \widetilde{\ell}_{k,s}\right)}$$

Sublinear regret bounds hold for suitable sequences (β_t) and (η_t):

$$t\eta_t \longrightarrow 0 \qquad \text{and} \qquad \eta_t \sum_{s \leqslant t} \beta_s \longrightarrow 0$$

(We often take $\beta_s = \square / s^2$ in the experimental studies.)

First study, continued

Forecasting of the air quality

Oracles
(RMSE of the experts and of fixed combinations thereof)

Uniform mean	Best expert	Best **p**
24.41	22.43	21.45

Semi-sequential strategies
(RMSE of the strategies tuned with best parameters in hindsight)

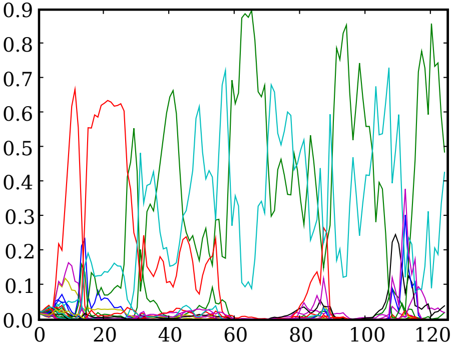
Original version	Moving sums ($H = 83$)	Discounts ($\beta_s = 1/s^2$)
21.47	21.37	21.31

Fully sequential strategies
(RMSE of the original version of the strategy)

Best parameter	Data-driven η_t
21.47	21.77

Our strategies do not focus on a single expert. We knew it from the numerical performance.

But actually, the weights associated with the experts change quickly and significantly over time and do not converge (which illustrates in passing that the performance of the considered experts varies over time).



Convex weights output by the (original) strategy with best parameter η in hindsight

A strategy to pick linear weights

It will ring a bell!

Linear combinations: Ridge regression (and the LASSO?)

The ridge regression was introduced in the 70s by Hoerl and Kennard; it was intensively studied since then in a stochastic setting.

We consider the case where $\mathcal{C} \subseteq \mathbb{R}$ and $\ell(x, y) = (x - y)^2$.

The ridge regression resorts to linear combinations of the experts:

$$\mathbf{v}_t \in \operatorname*{arg\,min}_{\mathbf{u} \in \mathbb{R}^N} \left\{ \lambda \, \|\mathbf{u}\|_2^2 + \sum_{s=1}^{t-1} \left(y_s - \sum_{j=1}^{N} u_j \, f_{j,s} \right)^2 \right\}$$

for some regularization parameter $\lambda > 0$.

It also exhibits a sublinear regret against individual sequences.

We do not know any such regret bounds for the LASSO yet.

Theorem. Consider a bound $B > 0$.

For all $\lambda > 0$, for all individual sequences of observations $y_t \in [-B, B]$ and of expert predictions $f_{j,t} \in [-B, B]$, for all $\mathbf{u} \in \mathbb{R}^N$,

$$\sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} v_{j,t}\, f_{j,t},\, y_t\right) - \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} u_j\, f_{j,t},\, y_t\right) \leqslant \lambda \|\mathbf{u}\|_2^2 + 2NB^2\left(1 + \frac{NTB^2}{\lambda}\right) \ln\left(1 + \frac{TB^2}{N\lambda}\right)$$

λ of the order of $1/\sqrt{T}$ is thus a good theoretical choice and leads to $O\big(\sqrt{T}\,\ln T\big)$ regret bounds.

Time-varying or data-driven parameters λ_t can be considered (both for theoretical bounds or for the sake of practical performance).

References: Vovk '01; Azoury and Warmuth '01; Gerchinovitz '11

The interest of this method is that it can compensate for biases (in either direction) as the weights do not need to sum up to 1.

Even better, we can/should use it as a pre-treatment on each single expert and

- turn it into a modified expert with predictions $\gamma_t \, f_{j,t}$,
- performing on average almost as well as the best expert of the form $\gamma \, f_{j,t}$ for some constant $\gamma \in \mathbb{R}$.

This would improve greatly the predictions if there existed, for instance, an almost constant multiplicative bias of $1/\gamma$.

First study, continued

Forecasting of the air quality

In our application to the prediction of air quality, we have extra sums over the stations of the network.

E.g., the ridge regression is defined as

$$\mathbf{v}_t \in \operatorname*{arg\,min}_{\mathbf{u}\in\mathbb{R}^N} \left\{ \lambda \|\mathbf{u}\|_2^2 + \sum_{\tau=1}^{t-1} \sum_{s\in\mathcal{S}_\tau} \left(y_\tau^s - \sum_{j=1}^{N} u_j\, f_{j,\tau}^s \right)^2 \right\}$$

One can show that $O\left(\sqrt{T}\ \ln T\right)$ regret bounds are still preserved.

The experts are indeed improved via the ridge pre-treatment. We illustrate this on the worst and best experts.

Original	Pre-treated	Original	Pre-treated
35.79	24.78	22.43	21.66

Oracles
(RMSE of the experts and of fixed combinations thereof)

Uniform mean	Best expert	Best **p**	Best **u**
24.41	22.43	21.45	19.24

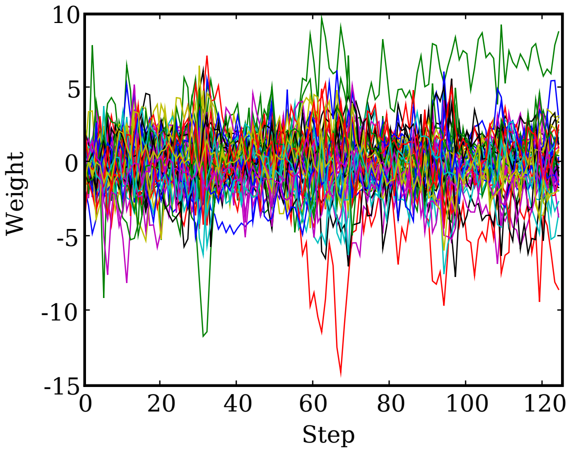
Semi-sequential ridge regression
(RMSE of the strategies tuned with best parameters in hindsight)

Original version	Moving sums ($H = 45$)	Discounts ($\beta_s = 100/s^2$)
20.77	20.03	19.45

Fully sequential ridge regression
(RMSE of the original version of the strategy)

Best parameter	Data-driven η_t
20.77	20.81

Our strategies do not focus on a single expert and the weights associated with the experts do not converge. [...]



Linear weights output by the (discounted) version of the ridge regression

Methodological summary

Methodological summary

1. Build the N experts (possibly on a training data set) and pick another data set for the evaluation of our methods, with T instances;
2. Compute some benchmarks and some reference oracles;
3. Evaluate our strategies when run with fixed parameters (i.e., with the best parameters in hindsight);
4. The performance of interest is actually the one of the data-driven meta-strategies.

We typically expect $T \geqslant 5N$ (or even $T \geqslant 10N$).

Hope arises when the oracles are 10% or 20% better than the methods used so far (e.g., the best expert when the latter is known in advance).

This usually requires the experts to be as different as possible.

On some data sets, the convex oracle does not improve much upon the best expert,

$$\min_{i=1,\ldots,N} \sum_{t=1}^{T} \ell(f_{i,t}, y_t) = \min_{\mathbf{q}} \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} q_j \, f_{j,t}, \; y_t\right)$$

In this case, one does not need to resort to the gradient trick.

It suffices to bound the regret with respect to the best expert,

$$\sum_{t=1}^{T} \ell\left(\sum_{j=1}^{N} p_{j,t} \, f_{j,t}, \; y_t\right) - \min_{i=1,\ldots,N} \sum_{t=1}^{T} \ell(f_{i,t}, y_t)$$
$$\leqslant \sum_{t=1}^{T} \sum_{j=1}^{N} p_{j,t} \widetilde{\ell}_{j,t} - \min_{i=1,\ldots,N} \sum_{t=1}^{T} \widetilde{\ell}_{i,t}$$

where the inequality follows by convexity and $\widetilde{\ell}_{j,t} = \ell(f_{j,t}, y_t)$.

Exponentially weighted averages (EWA) over the $\widetilde{\ell}_{j,t}$ can be applied.

Second study: Forecasting of exchange rates

Starting date: March 2012

Academic partner: Tomasz Michalski, HEC Paris

M.Sc. student involved over time:

- Christophe Amat (5 months in 2013; from Ecole Polytechnique)

Associated publication: In preparation

The goal is to predict monthly averages r_{t+1} of exchange rates based on few macro-economic indicators $x_{j,t}$ describing the state of the world:

- consumer price indexes (CPI);
- industrial production (Prod);
- monetary mass (Mon);
- required rates of return ("interest rates", 3R).

They will give rise to four experts.

The prediction horizon is 1-month ahead.

A classical stochastic modeling is

$$\ln r_{t+1} = \ln r_t + \sigma \left(W_{t+1} - W_t \right)$$

for some Brownian motion W.

It is considered difficult to improve on it (Meese and Rogoff '83).

It will give rise to the final expert ("random walk, RW").

We denote by r_t the averaged exchange rate of currency A with respect to currency B.

We focus on the log–variation $y_{t+1} = \Delta_{t+1} = \ln r_{t+1} - \ln r_t$.

The stochastic modeling suggested the prediction $f_{0,t+1} = 0$.

The economic theory indicates that a given macro-economic indicator $j \in \{1, 2, 3, 4\}$ can be used to forecast the exchange rate according to

$$\widehat{\Delta}_{j,t+1} = \ln x^B_{j,t} - \ln x^A_{j,t} \stackrel{\text{def}}{=} f_{j,t+1}$$

Using our methods we propose convex or linear combinations of the log–variations:

$$\widehat{\Delta}_{t+1} = \sum_{j=0}^{4} u_{j,t+1} f_{j,t+1} = \sum_{j=1}^{4} u_{j,t+1} f_{j,t+1}$$

Note: The fact that an expert suggests $f_{0,t+1}$ does matter!

The predicted log–variations $\widehat{\Delta}_t$ and exchange rates $\widehat{\Delta}_t + \ln r_{t-1}$ are evaluated via their common RMSE:

$$\widehat{\mathrm{RMSE}}_T = \sqrt{\frac{1}{T - t_0 + 1} \sum_{t=t_0}^{T} \left(\widehat{\Delta}_t - \Delta_t\right)^2} = \sqrt{\frac{1}{T - t_0 + 1} \sum_{t=t_0}^{T} \Big(\big(\widehat{\Delta}_t + \ln r_{t-1}\big) - \ln r_t \Big)^2}$$

where $t_0 = 30$ allows a short training period.

We apply two (families of) strategies:

– EWA (without a gradient trick), as it leads to interpretable weights;

– the ridge regression, as the regularization term should push in favor of expert 0 (the RW expert).

Some orders of magnitude for the prediction problems at hand are indicated below.

Time intervals	Every month
Period	April 1973 – May 2013
Time instances T	about 480
Number of experts N	$5 \ (= 1 + 4)$
GBP / USD	
Median of the Δ_t	1.48×10^{-2}
Maximum of the $\lvert\Delta_t\rvert$	11.08×10^{-2}
JPY / USD	
Median of the Δ_t	1.57×10^{-2}
Maximum of the $\lvert\Delta_t\rvert$	10.52×10^{-2}

Results for GBP / USD

Experts	RMSE	Oracle	RMSE
RW	2.47×10^{-2}	Best expert	2.47×10^{-2}
CPI	2.68×10^{-2}	Best $\mathbf{p}$	2.47×10^{-2}
3R	2.78×10^{-2}	Best $\mathbf{u}$	2.46×10^{-2}
Prod	2.66×10^{-2}		
Mon	2.75×10^{-2}		

vs.

EWA disc.	Semi-seq.	2.42×10^{-2}	
	Fully seq.	2.47×10^{-2}	(-0.2%)
Ridge disc.	Semi-seq.	2.34×10^{-2}	
	Fully seq.	2.37×10^{-2}	(-4.0%)

Results for JPY / USD

Experts	RMSE	Oracle	RMSE
RW	2.76×10^{-2}	Best expert	2.76×10^{-2}
CPI	2.89×10^{-2}	Best $\mathbf{p}$	2.75×10^{-2}
3R	2.96×10^{-2}	Best $\mathbf{u}$	2.74×10^{-2}
Prod	2.91×10^{-2}		
Mon	3.24×10^{-2}		

vs.

EWA disc.	Semi-seq.	2.73×10^{-2}	
	Fully seq.	2.75×10^{-2}	(-0.3%)
Ridge disc.	Semi-seq.	2.67×10^{-2}	
	Fully seq.	2.70×10^{-2}	(-2.2%)

Other empirical studies

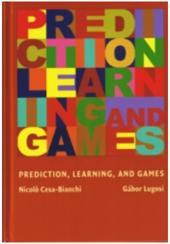
- Forecasting of the electricity consumption (EDF R&D)
- Forecasting of the production data of oil reservoirs (IFP–EN)

But time is over...

References

In case you're not bored to death (yet) by this topic!

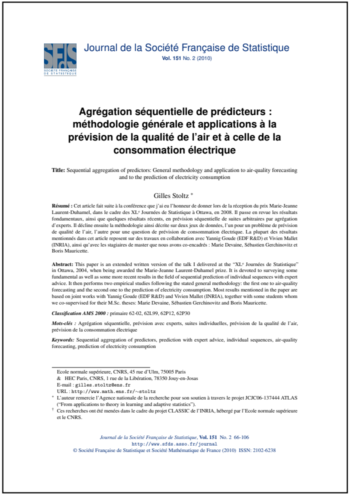
The so-called “red bible!”



Prediction, Learning, and Games

Nicolò Cesa-Bianchi et Gábor Lugosi

I published a survey paper (containing this talk!) one year ago in the Journal de la Société Française de Statistique



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Agrégation séquentielle de prédicteurs : méthodologie générale et applications à la prévision de la qualité de l'air et à celle de la consommation électrique

Title: Sequential aggregation of predictors: General methodology and application to air-quality forecasting and to the prediction of electricity consumption

Gilles Stoltz *

Résumé : Cet article fait suite à la conférence que j'ai eu l'honneur de donner lors de la réception du prix Marie-Jeanne Laurent-Duhamel, dans le cadre des XL^e Journées de Statistique à Ottawa, en 2008. Il passe en revue les résultats fondamentaux, ainsi que quelques résultats récents, en prévision séquentielle de suites arbitraires par agrégation d'experts. Il décline ensuite la méthodologie ainsi décrite sur deux jeux de données, l'un pour un problème de prévision de qualité de l'air, l'autre pour une question de prévision de consommation électrique. La plupart des résultats mentionnés dans cet article reposent sur des travaux en collaboration avec Yannig Goude (EDF R&D) et Vivien Mallet (INRIA), ainsi qu'avec les stagiaires de master que nous avons co-encadrés : Marie Devaine, Sébastien Gerchinovitz et Boris Mauricette.

Abstract: This paper is an extended written version of the talk I delivered at the "XL^e Journées de Statistique" in Ottawa, 2004, when being awarded the Marie-Jeanne Laurent-Duhamel prize. It is devoted to surveying some fundamental as well as some more recent results in the field of sequential prediction of individual sequences with expert advice. It then performs two empirical studies following the stated general methodology: the first one to air-quality forecasting and the second one to the prediction of electricity consumption. Most results mentioned in the paper are based on joint works with Yannig Goude (EDF R&D) and Vivien Mallet (INRIA), together with some students whom we co-supervised for their M.Sc. theses: Marie Devaine, Sébastien Gerchinovitz and Boris Mauricette.

Classification AMS 2000 : primaire 62-02, 62L99, 62P12, 62P30

Mots-clés : Agrégation séquentielle, prévision avec experts, suites individuelles, prévision de la qualité de l'air, prévision de la consommation électrique

Keywords: Sequential aggregation of predictors, prediction with expert advice, individual sequences, air-quality forecasting, prediction of electricity consumption

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Even better (or worse)—it is in French!