Correction for Exercise #9. [Thanks to Marin Billu!]

* The answer is no.

* Even my extension for Proof #2 was too quick. Indeed, I wrote \( E[X|G]\) and thus I implicitly assumed \( X \in L^1 \) (a minimal requirement for the existence of a conditional expectation, unless \( X > 0\), but as I will need \( X - E[X|G] \) I don’t want to run into a \( +\infty \) (or \( -\infty \)) trouble).

* So, assume \( X \in L^2\). Then my extension finally works and yields

\[
(x) \quad \ln E\left[ e^{s(X - E[X|G])} \right] \leq \frac{s^2}{8} (b-a)^2
\]

All is fine here, as \( X - E[X|G] \) is actually bounded, thus there is no surprise that \( e^{s(X - E[X|G])} \) is integrable:

\[
X \in L^1 \quad \text{implies} \quad e^X \in L^1
\]

since \( |X - G| \leq \max \left\{ \|a\|, \|b\| \right\} \) thus \( X - E[X|G] = (X-G) - E[X-G|G] \) is bounded.

\[
\frac{e^{s(X-G)}}{e^{s|G|}} \leq e^{s|G|}
\]

* If we really want the \( E[X|G]\) term outside of the \( \ln E[\cdot|G]\) in (x), we need to ensure that

\[
E\left[ e^{s(X - E[X|G])} | G\right] = E\left[ e^{sX} | G\right] e^{-s E[X|G]}
\]

In the first place, we need to ensure that \( E[e^{sX} | G] \) is finitely valued (as the two other terms are). We are led to assume that \( e^{sX} \in L^2\).

We then get:

\[
(x*) \quad \ln E[e^{sX} | G]\) \leq \frac{s^2}{8} (b-a)^2
\]
What about \( \text{Proof } \#3 ? \)

We want to get \((*)\) under the assumption

\[
X \in l^2 \quad \text{and} \quad a \leq X - G \leq b
\]

We consider \( H = \left\{ \text{ } \right\} \) \( E \left[ e^{s(x - \mathbb{E}(X|G_j))} \right] \geq e^{\frac{s^2}{2} (b-a)^2} \) \( G_j \) \( j \in G \)

If \( P(H) \geq \alpha \), then \((*)\)

\[
E_H \left[ e^{s(x - \mathbb{E}(X|G_j))} \right] \geq e^{\frac{s^2}{2} (b-a)^2}
\]

but the question is whether Hoeffding's lemma in its unconditional form ensured that \( E_H \left[ e^{s(x - \mathbb{E}(X|G_j))} \right] \leq e^{\frac{s^2}{2} (b-a)^2} \) so that we have a contradiction.

It is the case: apply Hoeffding's lemma in its unconditional form to \( X' = Y - G \in [a,b] \) and \( \text{IF}_H \). We again heavily use \( X \in l^2 \) this \( G \in l^2 \) to get

\[
X = E[X|G_j] = X - E[X|G]
\]

and all is fine !

Again, to get \((***)\) from \((*)\) we require \( e^{sX} \in l^2 \), under which \((***)\) immediately follows from \((*)\).
Solution for Exercise #1.

- \[ \mathbb{E}[N_i(T)] \leq \min \left\{ \frac{8\ln T}{\Delta_i^2} + 2 \right\} \leq \sqrt{T \left( \frac{8\ln T}{\Delta_i^2} + 2 \right)} \]

Thus, \( \hat{R}_T = \sum_{i : \Delta_i \geq \sigma} \Delta_i \mathbb{E}[N_i(T)] \leq \sum_{i : \Delta_i \geq \sigma} \sqrt{T \left( \frac{8\ln T}{\Delta_i^2} + 2 \right)} \leq O(K \sqrt{T \ln T}) \)

Or a more direct approach:

\[ \hat{R}_T = \sum_{i : \Delta_i \geq \sigma} \sqrt{\frac{8\ln T}{\Delta_i^2}} \Delta_i \leq \sum_{i : \Delta_i \geq \sigma} \frac{\Delta_i \sqrt{T}}{\sqrt{\ln T}} \leq \sqrt{T \ln T} = O(K \sqrt{T \ln T}) \]

- Where did we fail? We used that \( \mathbb{E}[N_i(T)] \leq T \)

but in fact, a stronger statement holds:

\[ \sum_i \mathbb{E}[N_i(T)] = T \]

- The smarter approach is:

\[ \hat{R}_T = \sum_{i : \Delta_i \geq \sigma} \Delta_i \mathbb{E}[N_i(T)] \leq \sum_{i : \Delta_i \geq \sigma} \Delta_i \min \left\{ \mathbb{E}[N_i(T)], \frac{8\ln T}{\Delta_i^2} + 2 \right\} \leq \sqrt{8\ln T + 2} \sum_{i = 1}^{k} \sqrt{\mathbb{E}[N_i(T)]} \]

\[ \leq \sqrt{8\ln T + 2} \sum_{i = 1}^{k} \sqrt{\mathbb{E}[N_i(T)]} \leq \sqrt{8\ln T + 2} \sum_{i = 1}^{k} \sqrt{\mathbb{E}[N_i(T)]} \leq \sqrt{8\ln T + 2} \sum_{i = 1}^{k} \sqrt{\mathbb{E}[N_i(T)]} = T \]

\[ = \sqrt{KT (8\ln T + 2)} \]