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Correction for Exercise #0. [ Thanks to Marin Ballu ! ]

- The answer is no.
- Even my extension for Proof #2 was too quick.

  Indeed, I wrote $\mathbb{E}[X|\mathcal{G}]$ and thus I implicitely assumed $X \in \mathbb{L}^1$ (a minimal requirement for the existence of a conditional expectation, unless $X \geq 0$, but as I will need $X - \mathbb{E}[X|\mathcal{G}]$ I don't want to run into a $+\infty - (+\infty)$ trouble).
- So, assume $X \in \mathbb{L}^1$. Then my extension finally works and yields

$$(*) \qquad \ln \mathbb{E}\left[ e^{s(X - \mathbb{E}[X|\mathcal{G}])} \,\middle|\, \mathcal{G} \right] \leq \frac{s^2}{8}(b-a)^2$$

All is fine here, as $X - \mathbb{E}[X|\mathcal{G}]$ is actually bounded, thus there is no surprise that $e^{s(X - \mathbb{E}[X|\mathcal{G}])}$ is integrable:

$X \in \mathbb{L}^1$ implies $G \in \mathbb{L}^1$ since $|X - G| \leq \max\{|a|, |b|\}$

thus $X - \mathbb{E}[X|\mathcal{G}] = \underbrace{(X-G)}_{\in [a,b]} - \underbrace{\mathbb{E}[(X-G)|\mathcal{G}]}_{\in [a,b]}$ is bounded.

- If we really want the $\mathbb{E}[X|\mathcal{G}]$ term outside of the $\ln \mathbb{E}[\ \ |\mathcal{G}]$ in $(*)$, we need to ensure that

$$\mathbb{E}\left[ e^{s(X - \mathbb{E}[X|\mathcal{G}])} \,\middle|\, \mathcal{G} \right] = \mathbb{E}\left[ e^{sX} \,\middle|\, \mathcal{G} \right] e^{-s\,\mathbb{E}[X|\mathcal{G}]}$$

In the first place, we need to ensure that $\mathbb{E}[e^{sX} | \mathcal{G}]$ is finitely valued (as the two other terms are). We are led to assume that $e^{sX} \in \mathbb{L}^1$.

We then get: $(**) \qquad \ln \mathbb{E}[e^{sX} | \mathcal{G}] \leq s\, \mathbb{E}[X|\mathcal{G}] + \dfrac{s^2}{8}(b-a)^2$

- What about Proof #3?

We want to get $(*)$ under the assumptions

$$X \in \mathbb{L}^1 \quad \text{and} \quad a \leq X - G \leq b$$

We consider $H = \left\{ \mathbb{E}\left[ e^{s(X - \mathbb{E}[X|\mathcal{G}])} \,\middle|\, \mathcal{G} \right] > e^{s^2/8\,(b-a)^2} \right\} \in \mathcal{G}$

If $\mathbb{P}(H) > 0$, then $(\dots)$

$$\mathbb{E}_H\left[ e^{s(X - \mathbb{E}[X|\mathcal{G}])} \right] > e^{s^2/8\,(b-a)^2}$$

but the question is whether Hoeffding's lemma in its unconditional form ensured that $\mathbb{E}_H\left[ e^{s(X - \mathbb{E}[X|\mathcal{G}])} \right] \leq e^{s^2/8\,(b-a)^2}$ so that we have a contradiction

It is the case: apply Hoeffding's lemma in its unconditional form to $X' = X - G \in [a,b]$ and $\mathbb{P}_H$. We again heavily use $X \in \mathbb{L}^1$ thus $G \in \mathbb{L}^1$ to get

$$X' - \mathbb{E}[X' | \mathcal{G}] = X - \mathbb{E}[X|\mathcal{G}]$$

and all is fine!

Again, to get $(**)$ from $(*)$ we require $e^{sX} \in \mathbb{L}^1$, under which $(**)$ immediately follows from $(*)$.

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## Solution for Exercise #1.

$a,b \geq 0$: $\min\{a,b\} \leq \sqrt{ab}$

- $$\mathbb{E}[N_i(T)] \leq \min\left\{T, \frac{8\ln T}{\Delta_i^2}+2\right\} \leq \sqrt{T\left(\frac{8\ln T}{\Delta_i^2}+2\right)}$$

thus $$\overline{R}_T = \sum_{i:\Delta_i>0} \Delta_i \, \mathbb{E}[N_i(T)] \leq \sum_{i:\Delta_i>0} \sqrt{T(8\ln T + 2\Delta_i^2)} \leq O\left(K\sqrt{T\ln T}\right)$$

Or a more direct approach:

$$\overline{R}_T = \sum_{i:\Delta_i > \sqrt{\frac{8\ln T}{T}}} \underbrace{\left(2+\frac{8\ln T}{\Delta_i^2}\right)\Delta_i}_{<2+\sqrt{8T\ln T}} + \sum_{\substack{i:\Delta_i \leq \sqrt{\frac{8\ln T}{T}} \\ \text{and } \Delta_i>0}} \underbrace{\Delta_i T}_{<\sqrt{8T\ln T}} \leq K\left(2+\sqrt{8T\ln T}\right) = O\left(K\sqrt{T\ln T}\right)$$

- Where did we fail? We used that $\forall i$, $\mathbb{E}[N_i(T)] \leq T$

  but in fact, a stronger statement holds: $$\sum_i \mathbb{E}[N_i(T)] = T$$

- The smarter approach is:

$$\begin{aligned}
\overline{R}_T &= \sum_{i:\Delta_i>0} \Delta_i \, \mathbb{E}[N_i(T)] && \text{by the Proposition} \\
&\leq \sum_{i:\Delta_i>0} \Delta_i \min\left\{\mathbb{E}[N_i(T)], \frac{8\ln T}{\Delta_i^2}+2\right\} && \min\{a,b\} \leq \sqrt{ab} \\
&\leq \sum_{i:\Delta_i>0} \sqrt{\mathbb{E}[N_i(T)]\left(8\ln T + 2\underbrace{\Delta_i^2}_{\leq 1}\right)} \\
&\leq \sqrt{8\ln T+2} \sum_{i=1,\dots,K} \sqrt{\mathbb{E}[N_i(T)]} && \sqrt{\phantom{x}} \text{ is concave: for } u_1,\dots,u_K \geq 0,\ \frac{1}{K}\sum_j \sqrt{u_j} \leq \sqrt{\frac{1}{K}\sum u_j} \\
&\leq \sqrt{8\ln T+2}\sqrt{K \underbrace{\sum_{i=1}^{K} \mathbb{E}[N_i(T)]}_{=T}} \\
&= \sqrt{KT(8\ln T+2)}
\end{aligned}$$