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Correction for Exercise #1.

- Given that when $\mathbb{P} \ll \mathbb{Q}$, we have

$$\mathrm{KL}(\mathbb{P},\mathbb{Q}) = \int_\Omega \left(\frac{d\mathbb{P}}{d\mathbb{Q}} \ln \frac{d\mathbb{P}}{d\mathbb{Q}}\right) d\mathbb{Q} \qquad \text{by definition of KL}$$

$$= \int_\Omega \left(\ln \frac{d\mathbb{P}}{d\mathbb{Q}}\right) d\mathbb{P} \qquad \text{by definition of } \frac{d\mathbb{P}}{d\mathbb{Q}}$$

Also, by definition of the density functions: $d\mathbb{Q} = g\,d\nu$ and $d\mathbb{P} = f\,d\nu$

Thus, to get $\mathrm{KL}(\mathbb{P},\mathbb{Q}) = \int_\Omega \left(\frac{f}{g} \ln \frac{f}{g}\right) g\,d\nu = \int_\Omega \left(\ln \frac{f}{g}\right) f\,d\nu$

we only need to prove that $\frac{f}{g}$ is a density of $\mathbb{P}$ wrt $\mathbb{Q}$.

- To that end, we need to be careful with the event $E = \{g=0\}$

We have $\mathbb{Q}(E) = \int \mathbb{1}_E \, d\mathbb{Q} = \int \mathbb{1}_{\{g=0\}}\, g\,d\nu = 0$,
thus, by $\mathbb{P} \ll \mathbb{Q}$: $\mathbb{P}(E) = 0$ as well.

Therefore, for all $A \in \mathcal{F}$,

$$\mathbb{P}(A) = \mathbb{P}(A \cap E^c) = \int \mathbb{1}_A \mathbb{1}_{\{g>0\}} f\,d\nu$$

$$= \int \mathbb{1}_A \mathbb{1}_{\{g>0\}} \frac{f}{g}\, g\,d\nu$$

$$= \int \mathbb{1}_A \frac{f}{g} \underbrace{\left(\mathbb{1}_{\{g>0\}}\, g\right) d\nu}_{= g\,d\nu = d\mathbb{Q}}$$

here, we heavily used the conventions $0/0 = 0$ and $0 \times +\infty = 0$

Thus, $\mathbb{P}(A) = \int_\Omega \mathbb{1}_A \frac{f}{g}\, d\mathbb{Q}$.

<u>Exercise 2.</u> [Credits: Marin Ballu]

We prove that $K\mathbb{P} \ll L\mathbb{Q}$

$$\iff \begin{cases} \mathbb{P} \ll \mathbb{Q} \quad \text{and} \\ K(\omega,\cdot) \ll L(\omega,\cdot) \quad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega \end{cases}$$

holds as soon as Ω' is a topological space with a countable base (a "second-countable space) and $\mathcal{F}'$ is the Borel σ-algebra.

The property satisfied by Ω' means that there exist some countable collection $\mathcal{U} = (U_i)_{i \geqslant 1}$ of open sets of Ω' such that each open set V of Ω' can be written $V = \bigcup_{i : U_i \subseteq V} U_i$, ie, as a countable union of elements in $\mathcal{U}$.

This is, for instance, the case when Ω' is a separable metric space (like $\Omega' = \mathbb{R}^{\mathbb{N}}$). This assumption was satisfied in our application of the chain rule, where the space Ω' was $\Omega' = \mathbb{R} \times [0,1]$.

<u>Consequence of the equivalence:</u> Assumption (∗) is not needed, and maybe (∗∗) is not needed either.

- If $K\mathbb{P} \not\ll L\mathbb{Q}$ and $\mathbb{P} \not\ll \mathbb{Q}$, then the stated inequality is true as it is $+\infty = +\infty$
- If $K\mathbb{P} \not\ll L\mathbb{Q}$ and $\mathbb{P} \ll \mathbb{Q}$, then $\exists A$ s.t. $\mathbb{P}(A) > 0$ and $K(\omega,\cdot) \not\ll L(\omega,\cdot)$ for $\omega \in A$

  Not sure that $\omega \mapsto KL\big(K(\omega,\cdot), L(\omega,\cdot)\big)$ is $\mathcal{F}$-measurable, but we may define anyway
  $$\int KL\big(K(\omega,\cdot), L(\omega,\cdot)\big)\,d\mathbb{P}(\omega) = +\infty \quad \text{as a convention.}$$
  In which case, the equality to prove is also $+\infty = +\infty$ and it holds true.

- Thus, we may assume with no loss of generality that

  $K\mathbb{P} \ll L\mathbb{Q}$, with density $f(\omega,\omega')$

  $\mathbb{P} \ll \mathbb{Q}$, $h(\omega)$

  The $\mathbb{Q}$- (thus the $L\mathbb{Q}$-) probability of $\{h=0\}$ equals 0, as

$$Q\{h=0\} = \int \mathbb{1}_{\{h=0\}}\, dQ = \int \mathbb{1}_{\{h=0\}}\, h\, dP = 0$$

Thus $\tilde{g}(\omega,\omega') = \dfrac{f(\omega,\omega')}{h(\omega)}$ is LQ-a.s. well-defined and is $\mathcal{F}\otimes\mathcal{F}'$-measurable, as the ratio of two such functions.

Question: does this $\tilde{g}$ coincide with $g : (\omega,\omega') \longmapsto \dfrac{dK(\omega,\cdot)}{dL(\omega,\cdot)}(\omega')$?
That would prove the measurability of g, i.e., Assumption $(**)$!

<u>Proof of $\Leftarrow$</u> : [Holds actually <u>without</u> any assumption on Ω']

$$\mathcal{A} = \{ E \in \mathcal{F}\otimes\mathcal{F}' : \; KP(E) = 0 \text{ or } LQ(E) > 0 \}$$
$$\mathcal{A}' = \{ E \in \mathcal{F}\otimes\mathcal{F}' : \; E \in \mathcal{A} \text{ and } E^c \in \mathcal{A} \}$$

We prove that $\mathcal{A}' = \mathcal{F}\otimes\mathcal{F}'$, which will indeed show that

$\forall E \in \mathcal{F}\otimes\mathcal{F}'$, $LQ(E) = 0$ and $KP(E) > 0$ cannot simultaneously hold

that is : $\forall E \in \mathcal{F}\otimes\mathcal{F}'$, $LQ(E) = 0 \Rightarrow KP(E) = 0$

$KP \ll LQ$.

We do so by showing that

1) $\mathcal{A}'$ is a σ-algebra
2) That it contains $\mathcal{F}\times\mathcal{F}'$

a) if $A\times B \in \mathcal{F}\times\mathcal{F}'$ is such that $LQ(A\times B) = 0$ then $KP(A\times B) = 0$

b) if $A\times B \in \mathcal{F}\times\mathcal{F}'$ is such that $LQ((A\times B)^c) = 0$ then $KP((A\times B)^c) = 0$

In both cases, this proves that $A\times B \in \mathcal{A}$ or $(A\times B)^c \in \mathcal{A}$, thus $A\times B \in \mathcal{A}'$

1) $\mathcal{A}'$ is a σ-algebra

- $\Omega \times \Omega'$ and $\emptyset$ are in $\mathcal{A}$, thus both are also in $\mathcal{A}'$
- By definition of $\mathcal{A}'$, $E \in \mathcal{A}' \Leftrightarrow E^c \in \mathcal{A}'$
- Let $(E_k)_{k \in \mathbb{N}}$ be a family of events in $\mathcal{A}'$: all E_k belong to $\mathcal{A}$

thus: either $\exists k \mid LQ(E_k) > 0$ then $LQ\left(\bigcup_k E_k\right) > 0$ as well

or $\forall k$, $K\mathbb{P}(E_k) = 0$ then $K\mathbb{P}\left(\bigcup_k E_k\right) = 0$ as well

and $\bigcup_k E_k \in \mathcal{A}$ thus also belongs to $\mathcal{A}'$

2) a) Let $A \in \mathcal{F}$ and $B \in \mathcal{F}'$ such that $LQ(A \times B) = 0$

$$= \int \mathbb{1}_A(\omega)\, L(\omega, B)\, dQ(\omega)$$

thus $L(\omega, B) = 0$ for Q-as all $\omega \in \Omega$

thus also for $\mathbb{P}$-as all $\omega \in \Omega$
(given $\mathbb{P} \ll Q$)

Now, since $K(\omega, \cdot) \ll L(\omega, \cdot)$ for $\mathbb{P}$-as all $\omega \in \Omega$,
we get $K(\omega, B) = 0$ for $\mathbb{P}$-as all $\omega \in \Omega$

Finally, $$0 = \int \mathbb{1}_A(\omega)\, K(\omega, B)\, d\mathbb{P}(\omega) = K\mathbb{P}(A \times B)$$

b) $(A \times B)^c = A \times B^c \cup A^c \times B \cup A^c \times B^c$

Since $LQ\left((A \times B)^c\right) = 0$, we have $LQ(A \times B^c) = LQ(A^c \times B) = LQ(A^c \times B^c) = 0$

thus by 2a), $K\mathbb{P}(A \times B^c) = K\mathbb{P}(A^c \times B) = K\mathbb{P}(A^c \times B^c) = 0$

that is, $K\mathbb{P}\left((A \times B)^c\right) = 0$.

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Proof of $\Rightarrow$: It is immediate that $KP \ll LQ \Rightarrow P \ll Q$.

To show that $K(\omega, \cdot) \ll L(\omega, \cdot)$ for P-almost all $\omega \in \Omega$

it suffices to show that:

$\exists N \in \mathcal{F}$ with $P(N) = 0$ s.t. $\forall \omega \notin N$,

$\forall B \in \mathcal{F}$, $K(\omega, B) = 0$ or $L(\omega, B) > 0$

We will

1) show the desired property but only $\forall B \in \mathcal{U}$, where $\mathcal{U} = (U_i)_{i \geq 1}$ was described when recalling what a second-countable space is

2) then extend the property to $\forall B \in \mathcal{F}$ as desired.

1) $A_i = \{ L(\cdot, U_i) = 0 \} \in \mathcal{F}$

with $LQ(A_i \times U_i) = \int \mathbb{1}_{\{L(\omega, U_i) = 0\}} L(\omega, U_i) \, dQ(\omega) = 0$

thus $KP(A_i \times U_i) = 0 = \int \underbrace{\mathbb{1}_{\{L(\omega, U_i) = 0\}} K(\omega, U_i)}_{\geq 0} \, dP(\omega)$

which entails $\mathbb{1}_{\{L(\omega, U_i) = 0\}} K(\omega, U_i) = 0$ for P-almost all $\omega \in \Omega$

either $L(\omega, U_i) > 0$
or $K(\omega, U_i) = 0$

Id est: $P(N_i) = 0$ where $N_i = \{ L(\cdot, U_i) = 0$ and $K(\cdot, U_i) > 0 \}$

The exact same argument can be done for U_i^c:

$P(N_i') = 0$ where $N_i' = \{ L(\cdot, U_i^c) = 0$ and $K(\cdot, U_i^c) > 0 \}$

and we take $N = \bigcup_{i \geq 1} (N_i \cup N_i')$

2) Fix $\omega \in \Omega \setminus N$, define $\mathcal{B}_\omega = \{ B \in \mathcal{F}' : K(\omega, B) = 0$ or $L(\omega, B) > 0 \}$

$\mathcal{B}_\omega' = \{ B \in \mathcal{F}' : B \in \mathcal{B}_\omega$ and $B^c \in \mathcal{B}_\omega \}$

then $\mathcal{B}_\omega'$ is a σ-algebra (same proof as for $\mathcal{A}'$)

1) indicates that since $\omega \in \Omega \setminus N$, we have $U_i \in \mathcal{B}_\omega$ and $U_i^c \in \mathcal{B}_\omega$

thus $U_i \in \mathcal{B}_\omega'$

thus (by the second-countable space property and because σ-algebras are stable by countable unions), all open sets belong to $\mathcal{B}'_W$.

But we assumed that $\mathcal{F}'$ was the Borel σ-algebra (ie, the σ-algebra generated by open sets): $\mathcal{B}'_W = \mathcal{F}'$,

which concludes this proof.