Reminder :

Some parts of the proofs could be simplified/improved, and your feedback is welcome.

In particular, I am looking for:

(1) A much shorter and much natural proof of the variational formula for the Kinf

Idea: $\mathrm{KL}(\ ) = \sup \dots$ by the variational formula for the KL

thus $\mathrm{Kinf}(\ ) = \inf \sup \dots = \sup \inf \dots$ by Sion's lemma

↳ Is anyone able to work out those details?

(Working them out in a neat way would yield 18/20 without the need to take the exam.)

(3) A better treatment of the remainder terms throughout the proof, leading to a final

$$\frac{\ln T}{\mathrm{Kinf}(\nu_1, \mu^*)} - O(\ln \ln T),$$

which I know is the optimal bound.

There was also a (very difficult) improvement referred to as (2) in last week's lecture notes.

VARIATIONAL FORMULA.

Lemma: For all $\nu \in \mathcal{P}([0,1])$ and $\mu \in (0,1)$:

$$\mathrm{Kinf}(\nu,\mu) = \max_{d \in [0, 1/1-\mu]} \mathbb{E}_\nu[\ln(1-d(X-\mu))]$$

where $X \sim \nu$

where there exists d in the argmax with $\mathbb{E}_\nu\left[\frac{1}{1-d(X-\mu)}\right] \leq 1$

Proof: * We denote $H : d \in [0, \frac{1}{1-\mu}) \mapsto \mathbb{E}_\nu[\ln(1-d(X-\mu))]$

H is well defined at least on $[0,\frac{1}{1-\mu})$ as: $0 < 1-d(1-\mu) \leq 1-d(X-\mu) \leq 1+d\mu$ (*)

For $d = 1/1-\mu$: the expectation is well defined but may $= -\infty$, thus we only define H on $[0, 1/1-\mu)$.

* It suffices to deal with the case $\mathbb{E}(\nu) < \mu$; indeed, when $\mathbb{E}(\nu) \geq \mu$, then $\underset{=0}{\mathrm{Kinf}(\nu,\mu-\varepsilon)} \nearrow_{\varepsilon \to 0} \mathrm{Kinf}(\nu,\mu) = 0$ while by Jensen's inequality, $\forall d \in (0, \frac{1}{1-\mu}]$, $\mathbb{E}_\nu[\ln(1-d(X-\mu))] \leq \ln \mathbb{E}_\nu[1-d(X-\mu)] = \ln(1-d(\underbrace{\mathbb{E}(\nu)-\mu}_{\geq 0})) \leq 0$

so that $\max_{d \in [0,1/1-\mu]} \mathbb{E}_\nu[\ln(1-d(X-\mu))] = [\text{value at } d=0] = 0$

* Let's build an intuition: because of (*), we can differentiate under the $\mathbb{E}$ sign for $d \in [0, 1/1-\mu)$:

$$H'(d) = \mathbb{E}_\nu\left[\frac{-(X-\mu)}{1-d(X-\mu)}\right] \qquad H''(d) = \mathbb{E}_\nu\left[\frac{-(X-\mu)^2}{(1-d(X-\mu))^2}\right] \leq 0$$

for $d > 0$:

$$= \frac{1}{d}\left(1 - \mathbb{E}_\nu\left[\frac{1}{1-d(X-\mu)}\right]\right)$$

H is concave on $[0, 1/1-\mu)$

$H'(d) = 0 \iff \mathbb{E}_\nu\left[\frac{1}{1-d(X-\mu)}\right] = 1$ for $d \in (0, 1/1-\mu)$

while $H'(0) = \mu - \mathbb{E}(\nu)$ (case excluded)

$H(0) = 0 \iff \mathbb{E}(\nu) = \mu$

By concavity and differentiability: if $\exists d^* \in (0, 1/1-\mu)$ s.t. $H'(d^*) = 0$ then $\sup_{d \in [0,1/1-\mu)} H(d) = H(d^*) = \max_{d \in [0,1/1-\mu)} H(d)$.

* In all cases, $H(d) \xrightarrow[d \to 1/1-\mu]{} \mathbb{E}[\ln(1-\frac{X-\mu}{1-\mu})] = \mathbb{E}[\ln(\frac{1-X}{1-\mu})] \in [-\infty, \ln\frac{1}{1-\mu})$

Indeed:

$$H(d) = \mathbb{E}_\nu\left[\ln\left(1-d(X-\mu)\right)\mathbb{1}_{\{X>\mu\}}\right] + \mathbb{E}_\nu\left[\ln\left(1-d(X-\mu)\right)\mathbb{1}_{\{X\leq\mu\}}\right]$$

↓ by monotone convergence

$$\mathbb{E}_\nu\left[\ln\left(1-\frac{1}{1-\mu}(X-\mu)\right)\times\mathbb{1}_{\{X>\mu\}}\right] \in [-\infty, +\infty)$$

↓ by dominated convergence

$$\mathbb{E}_\nu\left[\ln\left(1-\frac{1}{1-\mu}(X-\mu)\right)\times\mathbb{1}_{\{X\leq\mu\}}\right] \in \mathbb{R}$$

Thus, in all cases, $\sup_{d\in[0,1/1-\mu)} H(d) = \sup_{d\in[0,1/1-\mu]} H(d)$.

Note: we will distinguish two cases but they may be put under the same umbrella. They really use the same techniques and approaches. Maybe everything boils down to seeing whether $\nu[1] = 0$ or $\nu[1] > 0$, which would be a more natural distinction of cases.

Structure of the proofs:

- exhibit a distribution ν' such that $KL(\nu,\nu') = \sup_{d\in[0,1/1-\mu)} H(d)$
- show that $E(\nu') \geq \mu$ thus $K_{\inf}(\nu,\mu) \leq KL(\nu,\nu')$
- prove that in addition, $K_{\inf}(\nu,\mu) \geq KL(\nu,\nu')$

* First case : $\exists d^* \in (0, \frac{1}{1-\mu}) \mid \mathbb{E}_\nu\left[\frac{1}{1-d^*(X-\mu)}\right] = 1$

let $\nu' \ll \nu$ s.t. $\frac{d\nu'}{d\nu}(x) = \frac{1}{1-d^*(x-\mu)}$, ν' is a probability distribution over $[0,1]$

$$H'(d^*) = 0 = \mathbb{E}_\nu\left[\frac{-(X-\mu)}{1-d^*(X-\mu)}\right] \underset{X\sim\nu}{=} \int -(x-\mu)\frac{d\nu'}{d\nu}(x)\, d\nu(x) = \mu - \mathbb{E}(\nu')$$

$d\nu'(x)$

that is, $\mathbb{E}(\nu') = \mu$

$\forall \varepsilon \in (0,\mu)$, $\quad Kinf(\nu, \mu-\varepsilon) \leq KL(\nu,\nu')$

thus by left-continuity of Kinf, we get:

$$Kinf(\nu,\mu) \leq KL(\nu,\nu')$$

Now, $\frac{d\nu'}{d\nu}(x) \in (0,+\infty)$ for all $x \in [0,1]$,

thus $\nu \ll \nu'$ with $\frac{d\nu}{d\nu'} = \left(\frac{d\nu'}{d\nu}\right)^{-1}$;

take $B \in \mathcal{B}([0,1])$,

$$\int \left(\frac{d\nu'}{d\nu}\right)^{-1} 1_B\, d\nu' = \int \left(\frac{d\nu'}{d\nu}\right)^{-1} \frac{d\nu'}{d\nu}\, 1_B\, d\nu = \nu(B)$$

($\left(\frac{d\nu'}{d\nu}\right)^{-1}\frac{d\nu'}{d\nu} = 1$)

Therefore,

$$KL(\nu,\nu') \underset{\text{def of KL}}{=} \mathbb{E}_\nu\left[\ln\left(\frac{d\nu}{d\nu'}(X)\right)\right] = \mathbb{E}_\nu\left[\ln\left(1-d^*(X-\mu)\right)\right]$$

$$= H(d^*) = \max_{d \in [0, 1/1-\mu)} \mathbb{E}_\nu\left[\ln(1-d(X-\mu))\right]$$

$$= \max_{d \in [0, 1/1-\mu]} \text{"}\cdots\text{"}$$

ie, $Kinf(\nu,\mu) \geq KL(\nu,\nu')$

For the reverse inequality, let $\nu'' \in \mathcal{P}([0,1])$ s.t. $\nu'' \gg \nu$ and $\mathbb{E}(\nu'') > \mu$.

We will show that $KL(\nu,\nu'') \geq KL(\nu,\nu')$, which entails the desired inequality by taking the infimum over these ν''!

Indeed, we assume with no loss of generality that

$KL(\nu,\nu'') < +\infty$ so that $\ln \frac{d\nu}{d\nu''} \in L^1(\nu)$

We also had (by Jensen's inequality) $KL(\nu,\nu') < +\infty$ so that $\ln \frac{d\nu}{d\nu'} \in L^1(\nu)$

Indeed,

$$\int \left|\ln \frac{d\nu}{d\nu'}\right| d\nu = \int \left(\ln \frac{d\nu}{d\nu'}\right) \mathbb{1}_{\frac{d\nu}{d\nu'} \geq 1} \, d\nu + \int \left(-\ln \frac{d\nu}{d\nu'}\right) \mathbb{1}_{\frac{d\nu}{d\nu'} < 1} \, d\nu$$

integrals of non negative functions are always well-defined!

as $\nu \ll \nu'$

$$= \int \left(\frac{d\nu}{d\nu'} \ln \frac{d\nu}{d\nu'}\right) \mathbb{1}_{\frac{d\nu}{d\nu'} \geq 1} \, d\nu' + \int \left(-\frac{d\nu}{d\nu'} \ln \frac{d\nu}{d\nu'}\right) \mathbb{1}_{\frac{d\nu}{d\nu'} < 1} \, d\nu'$$

$x \mapsto x \ln x$ is bounded from below, by $c \in \mathbb{R}$

$$= \int \left(\frac{d\nu}{d\nu'} \ln \frac{d\nu}{d\nu'}\right) d\nu' + 2 \int \left(-\frac{d\nu}{d\nu'} \ln \frac{d\nu}{d\nu'}\right) \mathbb{1}_{\frac{d\nu}{d\nu'} < 1} \, d\nu'$$

$\mathrm{KL}(\nu, \nu') < +\infty$ — well-defined and $\leq -2c$

Thus, $\left|\ln \frac{d\nu}{d\nu'}\right| \in L^1(\nu)$ and similarly, $\left|\ln \frac{d\nu}{d\nu'}\right| \in L^1(\nu')$.

We go back to the proof of

$$\mathrm{KL}(\nu, \nu'') \geq \mathrm{KL}(\nu, \nu')$$

on the next page.

$$-KL(\nu,\nu'') + KL(\nu,\nu') = -\int \ln\left(\frac{d\nu}{d\nu''}\right) d\nu + \int \ln\left(\frac{d\nu}{d\nu'}\right) d\nu$$

because all integrands are in $L^1(\nu)$

$$= \int \left(-\ln\left(\frac{d\nu}{d\nu''}(x)\right) + \ln\left(1 - d^*(x-\mu)\right)\right) d\nu(x)$$

as already used often, $\frac{d\nu}{d\nu''} > 0$ ν-as (see below)

$$= \int \ln\left(\left(1 - d^*(x-\mu)\right) \frac{1}{\frac{d\nu}{d\nu''}(x)} \right) d\nu(x)$$

Jensen's inequality for $\ln$

$$\leq \ln \int \left(\left(1 - d^*(x-\mu)\right) \frac{1}{\frac{d\nu}{d\nu''}(x)} \right) d\nu(x)$$

Now, we do not necessarily have ν'' absolutely continuous wrt ν, but we write by the Radon-Nykodym theorem: $\nu'' = \nu''_{ac} + \nu''_{sing}$ (positive measures) where $\nu''_{ac} \ll \nu$ and $\nu''_{sing} \perp \nu$

$\perp$ means that the two probability measures are singular wrt each other; in particular,

$$\forall A \in \mathcal{B}([0,1]),\quad \nu(A) > 0 \Rightarrow \nu''_{sing}(A) = 0$$

We have $\frac{d\nu}{d\nu''}$ is ν-a.s. > 0, thus ν''_{ac}-as as well, with $\frac{d\nu''_{ac}}{d\nu} = \frac{1}{\frac{d\nu}{d\nu''}}$ ν-a.s.

Indeed: for $B = \left\{\frac{d\nu}{d\nu''} = 0\right\}$

$\nu \ll \nu''$ implies $\nu(B) = \int \mathbb{1}_{\left\{\frac{d\nu}{d\nu''}=0\right\}} d\nu = \int \mathbb{1}_{\left\{\frac{d\nu}{d\nu''}=0\right\}} \frac{d\nu}{d\nu''} d\nu'' = 0$ ($\frac{d\nu}{d\nu''} = 0$)

Now, for all $B \in \mathcal{B}([0,1])$,

$$\int \mathbb{1}_B \frac{1}{\left(\frac{d\nu}{d\nu''}\right)} d\nu \overset{\nu \ll \nu''}{=} \int \mathbb{1}_B \frac{1}{\left(\frac{d\nu}{d\nu''}\right)} \frac{d\nu}{d\nu''} d\nu''$$

not necessarily ν''-as !!!
$= 1$
it is equal to 1 on $\left\{\frac{d\nu}{d\nu''} > 0\right\}$
and equal to 0 on $\left\{\frac{d\nu}{d\nu''} = 0\right\}$
but we only know $\nu''\left\{\frac{d\nu}{d\nu''} = 0\right\} = 0$

Let's thus re-try with some more care:

Using that $\nu\left\{\frac{dr}{dr''} > 0\right\} = 1$ (and thus, $\nu''_{sing}\left\{\frac{dr}{dr''} > 0\right\} = 0$):

$$\int \mathbb{1}_B \frac{1}{\left(\frac{dr}{dr''}\right)} dr = \int \mathbb{1}_B \mathbb{1}_{\left\{\frac{dr}{dr''} > 0\right\}} \frac{1}{\left(\frac{dr}{dr''}\right)} dr$$

$$= \int \mathbb{1}_B \underbrace{\mathbb{1}_{\left\{\frac{dr}{dr''} > 0\right\}} \frac{1}{\left(\frac{dr}{dr''}\right)} \frac{dr}{dr''}}_{= \mathbb{1}_{\left\{\frac{dr}{dr''} > 0\right\}}} dr'' \qquad r \ll r''$$

$$= r''\left(B \cap \left\{\frac{dr}{dr''} > 0\right\}\right)$$

$$= \underbrace{r''_{ac}\left(B \cap \left\{\frac{dr}{dr''} > 0\right\}\right)}_{= r''_{ac}(B)} + \underbrace{r''_{sing}\left(B \cap \left\{\frac{dr}{dr''} > 0\right\}\right)}_{= 0}$$

as by absolute continuity $r''_{ac} \ll r$,

$$r\left\{\frac{dr}{dr''} = 0\right\} = 0$$

implies $r''_{ac}\left\{\frac{dr}{dr''} = 0\right\} = 0$ as well.

We go back to our bound:

$$-KL(\nu,\nu'') + KL(\nu,\nu') \leq \ln \int_{[0,1]} (1-d^*(x-\mu)) \underbrace{\frac{d\nu''_{ac}}{d\nu}(x)}_{\widetilde{d\nu''_{ac}}(x)} d\nu(x)$$

as $1-d^*(x-\mu)>0$ given $0 \leq d^* < \frac{1}{1-\mu}$

$$\leq \ln \int_{[0,1]} (1-d^*(x-\mu)) \underbrace{(d\nu''_{ac}(x) + d\nu''_{sing}(x))}_{=d\nu''(x)}$$

$$= \ln\left(1 - d^*\underbrace{(E(\nu'')-\mu)}_{>0}\right) \leq \ln 1 = 0$$

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* Second case: $E(\nu) < \mu$ and $H'(d)$ is of the sign of $H'(0) = \mu - E(\nu) > 0$ for all $d \in (0, \frac{1}{1-\mu})$

That is, $\mathbb{E}_\nu\left[\frac{1}{1-d(X-\mu)}\right] < 1 \quad \forall d \in (0, \frac{1}{1-\mu})$

$\downarrow$ $d \to \frac{1}{1-\mu}$ by monotone convergence on $\{X \geq \mu\}$, by dominated convergence on $\{X<\mu\}$

$$\mathbb{E}_\nu\left[\frac{1}{1-\frac{1}{1-\mu}(X-\mu)}\right]$$

$$= \mathbb{E}_\nu\left[\frac{1-\mu}{1-X}\right] \leq 1$$

thus $\mathbb{E}_\nu\left[\frac{1-\mu}{1-X}\right] \in (0,1]$

In particular: $\nu\{1\} = 0$ (otherwise, the expectation right above would be $+\infty$ given that $\mu<1$)

In this case, H is increasing over $[0, \frac{1}{1-\mu})$, and we already explained why $H(d) \xrightarrow[d \geq \frac{1}{1-\mu}]{} \mathbb{E}_\nu\left[\ln\left(1-\frac{1}{1-\mu}(X-\mu)\right)\right]$

Then, $\sup_{d \in [0, \frac{1}{1-\mu}]} \mathbb{E}_\nu\left[\ln(1-d(X-\mu))\right] = \max_{d \in [0, \frac{1}{1-\mu}]} \mathbb{E}_\nu[\dots]$

$$= \mathbb{E}_\nu\left[\ln\left(1-\frac{1}{1-\mu}(X-\mu)\right)\right] = \mathbb{E}_\nu\left[\ln\left(\frac{1-X}{1-\mu}\right)\right]$$

We now exhibit $\nu' \in \mathcal{P}([0,1])$ with $E(\nu') = \mu$ and

$$\begin{cases} Kinf(\nu,\mu) = KL(\nu,\nu') & \leftsquigarrow \text{we prove } \geq \text{ and } \leq \\ KL(\nu,\nu') = \mathbb{E}_\nu\left[\ln\left(\frac{1-X}{1-\mu}\right)\right] \end{cases}$$

We indeed set $d\nu'(x) = \frac{1-\mu}{1-x}\, d\nu(x) + \underbrace{\left(1 - \mathbb{E}_\nu\left[\frac{1-\mu}{1-X}\right]\right)}_{\in [0,1)} d\delta_1(x)$ ($\{1$ excluded, $\{0$ possible)

$E(\nu') = E_\nu\left[(X-1+1)\frac{1-\mu}{1-X}\right]$

$= -(1-\mu)E_\nu\left[\mathbb{1}_{X\neq 1}\right] + \underbrace{\left(1 - E_\nu\left[\frac{1-\mu}{1-X}\right]\right)}_{=\mu} 1$

$\Rightarrow E(\nu') = \mu$, thus in particular then, by left-continuity of Kinf: $\mathrm{Kinf}(\nu, \mu-\varepsilon) \leq KL(\nu, \nu')$ $\mathrm{Kinf}(\nu, \mu) \leq KL(\nu, \nu')$.

Also, $d\nu'(x) = \underbrace{\frac{1-\mu}{1-x}\, d\nu(x)}_{d\nu'_{ac}(x)} + \underbrace{\left(1 - E[\,]\right) d\delta_1(x)}_{d\nu'_{sing}(x)}$ unique Radon-Nikodym decomposition of ν' wrt ν

As above, $\frac{d\nu}{d\nu'_{ac}}(x) = \frac{1}{\frac{d\nu'_{ac}}{d\nu}(x)} = \frac{1-x}{1-\mu}$ ν-as

We may set $\frac{d\nu}{d\nu'_{sing}}(1) = 0$ as $\nu'|_{\{1\}} = 0$

thus $\frac{d\nu}{d\nu'}(x) = \frac{1-x}{1-\mu}$ ν-as which entails

$KL(\nu, \nu') = \int \left(\ln \frac{d\nu}{d\nu'}\right) d\nu$

$= E_\nu\left[\ln\left(\frac{1-X}{1-\mu}\right)\right]$.

It only remains to show that $\mathrm{Kinf}(\nu, \mu) \geq KL(\nu, \nu')$.

Let ν'' be any distribution such that $\nu'' \in \mathcal{P}([0,1])$ and $E(\nu'') > \mu$.

We may assume with no loss of generality that $KL(\nu, \nu'') < +\infty$; in particular $\nu \ll \nu''$.

Then, with the same arguments as in the first case:

$KL(\nu, \nu') - KL(\nu, \nu'') = \int \left(\ln\left(\frac{d\nu}{d\nu'}\right) - \ln\left(\frac{d\nu}{d\nu''}\right)\right) d\nu$

$= \int \left(\ln\left(\frac{1-x}{1-\mu}\, \frac{d\nu''_{ac}}{d\nu}(x)\right)\right) d\nu(x)$ ($x \mapsto \frac{1-x}{1-\mu}$ ↑; ℓ is >0 ν-as and $\frac{1}{\frac{d\nu}{d\nu''}} = \frac{d\nu''_{ac}}{d\nu}$)

$\overset{\text{Jensen}}{\leq} \ln \int \underbrace{\frac{1-x}{1-\mu}}_{\geq 0} \frac{d\nu''_{ac}}{d\nu}(x)\, d\nu(x)$ ($\frac{d\nu''_{ac}}{d\nu}(x)\, d\nu(x) = d\nu''_{ac}(x)$)

$\leq \ln \int \frac{1-x}{1-\mu}\, d\nu''(x)$ (we may add $d\nu''_{sing}(x)$)

$= \ln \underbrace{\frac{1-E(\nu'')}{1-\mu}}_{<1} < 0$

Hence, $KL(\nu, \nu') < KL(\nu, \nu'')$

and the desired inequality $KL(\nu, \nu') \leq \mathrm{Kinf}(\nu, \mu)$.

Part 2. Control of the a term

Last week, we proved

$$\sum_{t=K}^{T-1} \mathbb{P}\Big\{ \mathrm{Kinf}\big(\hat{\nu}_a(t), \mu^* - \delta\big) \leq \frac{f(t)}{N_a(t)} \text{ and } I_{t+1} = a \Big\}$$

$$\leq \sum_{t=K}^{T-1} \sum_{\ell=1}^{n_b - 1} \mathbb{P}\big\{ N_a(t) = \ell \text{ and } I_{t+1} = a \big\} \qquad (*)$$

$$+ \sum_{t=K}^{T-1} \sum_{\ell=n_b}^{T-1} \mathbb{P}\Big\{ \mathrm{Kinf}\big(\hat{\nu}_a(t), \mu^*\big) \leq \mathrm{Kinf}\big(\nu_a, \mu^*\big) - \frac{\delta}{1-\mu^*} \Big\}$$

where $n_b = \left\lceil \dfrac{f(T)}{\mathrm{Kinf}(\nu_a, \mu^*) - \frac{2\delta}{1-\mu^*}} \right\rceil$

We use:

→ for all $\ell \in \{1, \dots n_b - 1\}$, there exists at most one $t \in \{K, \dots T-1\}$ such that $N_a(t) = \ell$ and $I_{t+1} = a$

is so that $\sum_{t=K}^{T-1} \sum_{\ell=1}^{n_b - 1} \mathbb{P}\{ N_a(t) = \ell \text{ and } I_{t+1} = a \} \leq n_b - 1$

→ by the variational formula, there exists $\tilde{d} \in [0, 1/1-\mu^*]$ s.t.

$$\begin{cases} \mathbb{E}_{\nu_a}\left[\dfrac{1}{1 - \tilde{d}(X - \mu^*)} \right] \leq 1 \quad \text{where } X \sim \nu_a \\ \mathrm{Kinf}\big(\nu_a, \mu^*\big) = \mathbb{E}_{\nu_a}\big[\ln\big(1 - \tilde{d}(X - \mu^*)\big) \big] \end{cases}$$

and also, $\mathrm{Kinf}\big(\hat{\nu}_a(t), \mu^*\big) \geq \dfrac{1}{N_a(t)} \sum_{s=1}^{t} \ln\big(1 - \tilde{d}(Y_s - \mu^*)\big) \mathbb{1}_{\{I_s = a\}}$

Therefore, the second term in $(*)$ is smaller than

$$\sum_{\ell=n_b}^{T-1} \Bigg(\sum_{t=K}^{T-1} \mathbb{P}\Big\{ \frac{1}{N_a(t)} \sum_{s=1}^{t} \ln\big(1 - \tilde{d}(Y_s - \mu^*)\big) \mathbb{1}_{\{I_s = a\}} \leq \mathbb{E}Z - \frac{\delta}{1-\mu^*} \text{ and } N_a(t) = \ell \text{ and } I_{t+1} = a \Big\} \Bigg)$$

where $\mathbb{E}Z$ = short-hand notation for $\mathbb{E}_{\nu_a}\big[\ln\big(1 - \tilde{d}(X - \mu^*)\big) \big]$ where $X \sim \nu_a$

To continue the bounding, we will use the following lemmas:

Lemma (one-sided Hoeffding's inequality, almost tailored to our needs):

Let $X_1, X_2, \dots$ be iid random variables such that:

$X_1 \in L^1$, $X_1 \leq a$ a.s. for some $a > 0$,

$\mathbb{E}[e^{-X_1}] \leq b < +\infty$ for some b.

Then, denoting $\gamma = \sqrt{e^a}(8b + a^2)$, we have:

$$\forall \varepsilon \in [0, \gamma/2], \quad \mathbb{P}\left\{ \frac{1}{T}\sum_{t=1}^{T} X_t - \mathbb{E}X_1 \leq -\varepsilon \right\} \leq e^{-T\varepsilon^2/2\gamma}$$

Lemma (extension to our setting of random number of summands):

Let $\tilde{d} \in [0, 1/1-\mu^*]$ and denote $Z = \ln(1 - \tilde{d}(Y^{(a)} - \mu^*))$ where $Y^{(a)} \sim \nu_a$.

If $\mathbb{E}_{\nu_a}[e^{-Z}] = \mathbb{E}_{\nu_a}\left[\frac{1}{1-\tilde{d}(Y^{(a)}-\mu^*)}\right] \leq 1$,

then (denoting $X_s = \ln(1-\tilde{d}(Y_s - \mu^*))$) we have, $\forall \varepsilon \in (0, \frac{9/2}{(1-\mu^*)^{5/2}})$ and $\forall \ell \geq 1$,

$$\sum_{t=1}^{T} \mathbb{P}\left\{ \left(\frac{1}{N_a(t)} \sum_{s=1}^{t} X_s \mathbb{1}_{\{I_s = a\}}\right) - \mathbb{E}Z \leq -\varepsilon \right.$$

$$\left. \text{and } N_a(t) = \ell \text{ and } I_{t+1} = a \right\}$$

$$\leq \exp\left(-\frac{\ell\varepsilon^2}{18}(1-\mu^*)^{5/2}\right)$$

We go back to the upper bound on the a term:

$$\leq n_0 - 1 + \sum_{\ell=n_0}^{T-1} \exp\left(-\ell\delta^2\sqrt{1-\mu^*}/18\right)$$

↓ by definition of n_0; by a geometric sum

$$\leq \frac{f(T)}{K_{\inf}(\nu_a, \mu^*) - 2\delta/1-\mu^*} + \frac{1}{1 - \exp(-\delta^2\sqrt{1-\mu^*}/18)}$$

and we substitute $f(T) = \ln T + \ln\ln T$.

Proof (iid version) We first apply Cramer-Chernoff with $x<0$, keeping in mind $\varepsilon>0$:

$$\mathbb{P}\left\{\frac{1}{T}\sum_{t=1}^{T} X_t - \mathbb{E}X_1 \leq -\varepsilon\right\} = \mathbb{P}\left\{x\sum_{t=1}^{T}(X_t - \mathbb{E}X_t) \geq -Tx\varepsilon\right\}$$
$$= \mathbb{P}\left\{\exp\left(x\sum_{t=1}^{T}(X_t - \mathbb{E}X_t)\right) \geq e^{-Tx\varepsilon}\right\}$$
$$\leq e^{Tx\varepsilon}\,\mathbb{E}\left[\exp\left(x\sum_{t=1}^{T}(X_t - \mathbb{E}X_t)\right)\right]$$
$$= e^{Tx\varepsilon}\left(\mathbb{E}\left[e^{x(X_1 - \mathbb{E}X_1)}\right]\right)^T.$$

As in the proof of the classical version of Hoeffding's lemma, we study

$$\Psi : x \mapsto \mathbb{E}\left[e^{xX_1}\right]$$

$\begin{cases} e^{X_1} \leq e^a \text{ thus } \Psi(1) \leq e^a < +\infty \\ \Psi(-1) = \mathbb{E}\left[e^{-X_1}\right] \leq b < +\infty \text{ by assumption} \end{cases}$

First: For which x is Ψ defined?

No matter whether $X_1 \geq 0$ or $X \leq 0$, we have

$$\forall x \in [-1,1], \quad e^{xX_1} \leq e^{-X_1} + e^{X_1} \in \mathbb{L}^1$$

Thus, Ψ is defined at least on $[-1,1]$

Second: Ψ is differentiable on $(-1,1)$

As $\forall x \in (-1,1) \quad \exists C_x \mid \quad X_1 e^{xX_1} \leq C_x\left(e^{-X_1} + e^{X_1}\right) \in \mathbb{L}^1$

(follows from $ue^{xu} = o\left(e^{|u|(|x|+\varepsilon)}\right)$ as $u \to +\infty$ and $u \to -\infty$)

with derivative $\Psi'(x) = \mathbb{E}\left[X_1 e^{xX_1}\right]$

Third: We can repeat the argument: Ψ is twice differentiable on $(-1,1)$, with second derivative $\Psi''(x) = \mathbb{E}\left[X_1^2 e^{xX_1}\right]$

Take $\varphi = \ln\Psi : x \in (-1,1) \mapsto \ln \mathbb{E}\left[e^{xX_1}\right]$; it is twice differentiable, with

$$\varphi'(x) = \mathbb{E}\left[X_1 e^{xX_1}\right] / \mathbb{E}\left[e^{xX_1}\right]$$

and

$$\varphi''(x) = \frac{\mathbb{E}\left[X_1^2 e^{xX_1}\right]\mathbb{E}\left[e^{xX_1}\right] - \left(\mathbb{E}\left[X_1 e^{xX_1}\right]\right)^2}{\left(\mathbb{E}\left[e^{xX_1}\right]\right)^2} \leq \frac{\mathbb{E}\left[X_1^2 e^{xX_1}\right]}{\mathbb{E}\left[e^{xX_1}\right]}$$

which we upper bound (and then, we will apply a second-order Taylor formula)

$-\infty \le X_1 \le a$ a.s. thus $\forall x<0$, $xa \le xX_1 \le +\infty$

First: $\forall u \in [-\infty, a]$ and $\forall x \in [-1/2, 0]$, $u^2 e^{xu} \le 8e^{-u} + a^2$

Indeed: if $u \ge 0$, $u^2 e^{xu} \le u^2 \le a^2$

if $u \le 0$, $u^2 \le 8e^{-u/2}$ and $xu \le -u/2$

thus $u^2 e^{xu} \le 8e^{-u/2+xu} \le 8e^{-u}$

the constant 8 is improvable, $16e^{-2} \le 2.2$ works

improvable $16e^{-2} \le 2.2$ works

Thus $\mathbb{E}[X_1^2 e^{xX_1}] \le 8\,\mathbb{E}[e^{-X_1}] + a^2 \le 8b + a^2$ for all $x \in [-1/2, 0]$

Second: $\mathbb{E}[e^{xX_1}] \ge e^{xa} \ge 1/\sqrt{e^a}$ for all $x \in [-1/2, 0]$

Finally, $\varphi''(x) \le \dfrac{\mathbb{E}[X_1^2 e^{xX_1}]}{\mathbb{E}[e^{xX_1}]} \le \sqrt{e^a}(8b + a^2) = \gamma$ $\forall x \in [-1/2, 0]$

Taylor's formula yields:

$\forall x \in [-1/2, 0]$, $\exists r_x \in [x, 0]$ | $\varphi(x) = \cancel{\varphi(0)} + x\underbrace{\varphi'(0)}_{\mathbb{E}X_1} + \dfrac{x^2}{2}\varphi''(r_x)$

That is, $\forall x \in [-1/2, 0]$, $\psi(x) = e^{\varphi(x)} \le e^{x\mathbb{E}X_1} \times \exp\left(\dfrac{\gamma}{2}x^2\right)$

Substituting in the first set of inequalities of the proof:

$$\mathbb{P}\left\{\frac{1}{T}\sum_{t=1}^{T} X_t - \mathbb{E}X_1 \le -\varepsilon\right\} \le \inf_{x \in (-1/2, 0)} \exp\left(Tx\varepsilon + T\frac{\gamma}{2}x^2\right)$$

minimum of $x \in (-\infty, 0) \mapsto x\varepsilon + \dfrac{\gamma}{2}x^2$ is achieved at

x^* s.t. $-\varepsilon = \gamma x^*$, $x^* = -\varepsilon/\gamma$

which by assumption lies in $[-1/2, 0]$

We get the stated upper bound:

$$\exp\left(Tx^*\varepsilon + T\frac{\gamma}{2}(x^*)^2\right) = \exp\left(-\frac{T\varepsilon^2}{\gamma} + \frac{\gamma}{2}\frac{T\varepsilon^2}{\gamma^2}\right)$$

$$= \exp\left(-T\varepsilon^2/2\gamma\right).$$

Proof (version with random number of summands)

$Y_t \in [a,b]$ and $d \geq 0$ Thus

$$\ln(1-d(1-\mu^*)) \leq X_t \leq \ln(1+d\mu^*) \leq \ln(1/1-\mu^*) \overset{\text{not.}}{=} a$$

given that $d \leq 1/1-\mu^*$

and also, by assumption, $\mathbb{E}[e^{-Z}] \leq 1 \overset{\text{not.}}{=} b$

We set $\gamma = \sqrt{e^a}\left(8b+a^2\right) = \frac{1}{\sqrt{1-\mu^*}}\left(8+\left(\ln(1/1-\mu^*)\right)^2\right)$.

Now, looking at the proof of our (one-sided) Hoeffding's lemma, it is clear that it goes through when replacing all expectations $\mathbb{E}$ by conditional expectations $\mathbb{E}[\,\cdot \mid \mathcal{G}_j]$ wrt some σ-algebra $\mathcal{G}_j$.

In particular, we can show that for all $t \geq 1$, denoting $\mathcal{F}_{t-1} = \sigma(Y_1 \dots Y_{t-1})$,

$$\mathbb{E}\left[\left(e^{x(X_t-\mathbb{E}Z)}\right)\mathbb{1}_{\{I_t=a\}} \mid \mathcal{F}_{t-1}\right] \leq \exp\left(\frac{\gamma}{2}x^2\right) \qquad \forall x \in [-1/2, 0]$$

By our trick $1 = \mathbb{1}_{\{I_t=a\}} + \mathbb{1}_{\{I_t \neq a\}}$, this inequality self-improves into

$$\mathbb{E}\left[e^{x(X_t-\mathbb{E}Z)}\mathbb{1}_{\{I_t=a\}} \mid \mathcal{F}_{t-1}\right] \leq \exp\left(\frac{\gamma}{2}x^2\,\mathbb{1}_{\{I_t=a\}}\right) \qquad \forall x \in [-1/2, 0]$$

We can then show that $\forall x \in [-1/2, 0]$,

$$\left(M_T\right)_{T\geq 1} = \left(\exp\left(x\sum_{t=1}^{T}(X_t-\mathbb{E}Z)\mathbb{1}_{\{I_t=a\}} - \frac{\gamma}{2}x^2 N_a(T)\right)\right)_{T\geq 1}$$

is a non-negative supermartingale.

Hence, Doob's maximal inequality applies:

$$\forall \beta > 0, \qquad \mathbb{P}\left\{\sup_{T\geq 1} M_T \geq \beta\right\} \leq \frac{\mathbb{E}[M_1]}{\beta} \leq \frac{1}{\beta}$$

The event of interest can be rewritten; for $x \in [-1/2, 0]$:

$$\sum_{t=1}^{T} \mathbb{P}\left\{\frac{1}{N_a(t)}\sum_{s=1}^{t} X_s \mathbb{1}_{\{I_s=a\}} - \mathbb{E}Z \leq -\varepsilon \text{ and } N_a(t)=\ell \text{ and } I_{t+1}=a\right\}$$

$$= \sum_{t=1}^{T} \mathbb{P}\left\{\sum_{s=1}^{t}(X_s-\mathbb{E}Z)\mathbb{1}_{\{I_s=a\}} \leq -\ell\varepsilon \text{ and } N_a(t)=\ell \text{ and } I_{t+1}=a\right\}$$

$$\leq \sum_{t=1}^{T} \mathbb{P}\left\{M_t \geq e^{-x\ell\varepsilon - \frac{\gamma}{2}x^2\ell} \text{ and } N_a(t)=\ell \text{ and } I_{t+1}=a\right\}$$

$$\leq \sum_{t=1}^{T} \mathbb{P}\left\{\sup_{G\geq 1} M_G \geq e^{-x\ell\varepsilon-\frac{\gamma}{2}x^2\ell} \text{ and } N_a(t)=\ell \text{ and } I_{t+1}=a\right\}$$

This last sum equals $\mathbb{P}\left\{ \sup_{G \geq 1} M_G \geq e^{-x\ell\varepsilon - \frac{\gamma}{2}x^2\ell} \right\}$

as there exists at most one t such that $N_t(t) = \ell$ and $I_{t+1} = a$ and as $\left\{ \sup_{G \geq 1} M_G \geq e^{-x\ell\varepsilon - \frac{\gamma}{2}x^2\ell} \right\}$ does not depend on t

We can bound it by Doob's inequality; we get the bound

$$\exp\left(\ell x\varepsilon + \frac{\gamma}{2} x^2 \ell \right) \leq \exp\left(\ell \left(x\varepsilon + \frac{\gamma'}{2} x^2 \right) \right)$$

to be minimized over $x \in [-1/2, 0]$

with $\gamma' = \frac{g}{(1-\mu^*)^{5/2}}$

$$\geq \frac{1}{\sqrt{1-\mu^*}} \left(\delta + \ln\left(\frac{1}{1-\mu^*} \right)^2 \right)$$ as $\ln x \leq x$ for $x > 0$

↳ pick $x = -\varepsilon/\gamma'$, which lies indeed in $[-1/2, 0]$ thanks to the assumption $\varepsilon < \frac{g/2}{(1-\mu^*)^{5/2}}$

↳ we get the bound

$$\exp\left(-\ell\varepsilon^2 / 2\gamma' \right)$$

Conclusion of the proof:

Let's collect all bounds together!

For $\mu^* \in (q_1)$, $\delta \in (0, \mu^*)$, and $T \geq K$:

a^* term, see Part 1 — a term, see Part 2

$$\mathbb{E}_\nu[N_a(T)] \leq 1 + \sum_{t=K}^{T+1} \mathbb{P}\{U_{a^*}(t) \leq \mu^* - \delta\} + \sum_{t=K}^{T+1} \mathbb{P}\left\{ K_{\inf}(\hat{\nu}_a(t), \mu^* - \delta) \leq \frac{f(t)}{N_a(t)} \right\} \text{ and } I_{t+1} = a^*$$

$$\leq 1 + \frac{2e^2}{(1-e^{-\delta^2/2})^2}\left(2 + \ln\ln T\right) + \frac{\ln T + \ln\ln T}{K_{\inf}(\nu_a, \mu^*) - 2\delta/1-\mu^*} + \frac{1}{1 - e^{-\delta^2\sqrt{1-\mu^*}/8}}$$

To be optimized over $\delta \in (0, \mu^*)$ to get a non-asymptotic bound.

Which requires some preliminary further bounding (which can probably be improved...).

One is: $\begin{cases} \frac{1}{1-e^{-x}} \leq 1 + \frac{2}{x} \\ \frac{1}{(1-e^{-x})^2} \leq 1 + \frac{2}{x^2} \end{cases} \quad \forall x > 0$ (they convey the correct orders of magnitude)

The other is $\frac{1}{1-x} \leq 1 + 2x \quad \forall x \in (0, 1/2]$

We get, for $\delta \in (0, \mu^*)$ such that $\frac{2\delta}{(1-\mu^*)\, \mathrm{Kinf}(\nu_a, \mu^*)} \leq 1/2$ [condition on δ]

$$\mathbb{E}_\nu[N_a(T)] \leq \frac{\ln T}{\mathrm{Kinf}(\nu_a, \mu^*)} + \frac{4\delta \ln T}{(1-\mu^*)\left(\mathrm{Kinf}(\nu_a, \mu^*)\right)^2}$$

$$+ \frac{\ln \ln T}{\mathrm{Kinf}(\nu_a, \mu^*)} + \frac{4\delta \ln \ln T}{(1-\mu^*)\left(\mathrm{Kinf}(\nu_a, \mu^*)\right)^2}$$

$$+ 1 + 2e^2\left(1 + \frac{8}{\delta^4}\right)(2 + \ln \ln T)$$

$$+ 1 + \frac{36}{\delta^2 \sqrt{1-\mu^*}}$$

We need to balance the terms $\delta \ln T$ and $1/\delta^4$, and will thus take δ of the order of $(\ln T)^{-1/5}$, e.g.,

$$\delta = \frac{(1-\mu^*)\, \mathrm{Kinf}(\nu_a, \mu^*)}{4} \underbrace{(\ln T)^{-1/5}}_{\leq 1 \text{ for } T \geq 3}$$

so that the stated condition on δ is satisfied for $T \geq 3$.

We can now write a final and clean bound, of the form:

$$\mathbb{E}_\nu[N_a(T)] \leq \frac{\ln T}{\mathrm{Kinf}(\nu_a, \mu^*)} + O\left((\ln T)^{4/5} \ln \ln T\right)$$