 Lecture 1 (corrections) Lecture 1 (corrections)

Various feedback on the course:

By Ludovic Schwartz: a convex function $f : \mathcal{X} \to \mathbb{R}$ (or even $[0,1]$) does not necessarily achieve its infimum,

hence we safely and rightfully considered p^*_δ as an approximate arg inf in the last proof of Lecture 1

Ex: $f(x) = x^2 + \mathbb{1}_{\{x=0\}}$, infimum equal to 0, never achieved

(take $f(x) = x^2 + \mathbb{1}_{\{x=0\}}$ for an exp-concave example)

By Hassan Saber: in the last proof, it is easy to directly optimize over $\varepsilon \in (0,1)$ the obtained bound:

$$B(\varepsilon) = T \ln(1-\varepsilon) + (N-1) \ln \varepsilon \qquad B'(\varepsilon) = \frac{-T}{1-\varepsilon} + \frac{N-1}{\varepsilon}$$

$B''(\varepsilon) < 0$ thus the ε^* s.t. $B'(\varepsilon^*) = 0$ is the global maximum of B

We have $B'(\varepsilon^*) = 0 \iff \dfrac{\varepsilon^*}{N-1} = \dfrac{1}{T} - \dfrac{\varepsilon^*}{T} \iff \varepsilon^* = \dfrac{1/T}{1/T + 1/(N-1)} = \dfrac{N-1}{T+N-1}$

We thus get the sharper final bound

$$\frac{1}{\eta}\left(T \ln\left(\frac{T+N-1}{T}\right) + (N-1)\ln\left(\frac{T+N-1}{N-1}\right)\right) = \frac{T+N-1}{\eta} H\left(\frac{N-1}{T+N-1}\right)$$

$$= \frac{1}{\eta}\left(\underbrace{(N-1)\ln\left(1+\frac{N-1}{T}\right)}_{\leqslant (N-1)^2/T} + (N-1)\ln\left(1+\frac{T}{N-1}\right)\right)$$

where $H(x) = x \ln x + (1-x)\ln(1-x)$ is the binary entropy

* However * both bounds are $\sim \dfrac{N-1}{\eta} \ln T$ as $T \to +\infty$.

Exercise #1 / Various considerations around the notion of regret.

1) ($N = 2$ is enough)

Time	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	etc.
ℓ_{1t}	1	0	1	0	1	0	
$\sum_{s=1}^{t-1} \ell_{1s}$		1	1	2	2	3	
ℓ_{2t}	$1/2$	1	0	1	0	1	
$\sum_{s=1}^{t-1} \ell_{2s}$		$1/2$	$1+1/2$	$1+1/2$	$2+1/2$	$2+1/2$	
Leader over $s=1,\dots,t-1$		2	1	2	1	2	
p_{1t}	$1/2$	0	1	0	1	0	
p_{2t}	$1/2$	1	0	1	0	1	
$\sum_{j=1}^{2} p_{jt}\ell_{jt}$	$\frac{3}{4}$	1	1	1	1	1	
$\sum_{s=1}^{t} \sum_j p_{jt}\ell_{jt}$	$\frac{3}{4}$	$1+\frac{3}{4}$	$2+\frac{3}{4}$	$3+\frac{3}{4}$	$4+\frac{3}{4}$	$5+\frac{3}{4}$	

Follow the leader is a strategy that fails miserably $\longrightarrow$ EWA is a smoothed alternative.

We have
$$\sum_{t=1}^{T} \sum_{j=1}^{2} p_{jt}\ell_{jt} = T-1 + \frac{3}{4}$$

$$\sum_{t=1}^{T} \ell_{1t} = \begin{cases} T/2 & \text{if } T \text{ even} \\ \frac{T-1}{2} & \text{if } T \text{ odd} \end{cases}$$ (lower integer part)

$$\sum_{t=1}^{T} \ell_{2t} = \begin{cases} (T-1)/2 + 1/2 & \text{if } T \text{ odd} \\ T/2 - 1 + 1/2 & \text{if } T \text{ even} \end{cases}$$

$$\min\left\{ \sum_{t=1}^{T} \ell_{1t},\ \sum_{t=1}^{T} \ell_{2t} \right\} \leqslant \frac{T}{2}$$

$$R_T \geqslant T-1 + \frac{3}{4} - T/2 = T/2 - 1/4 \neq o(T)$$

2) It suffices to consider only the sequences in $\{0,1\}^{NT}$;

For any given strategy, we denote for $t \geqslant 2$:

$$k_t^* \in \operatorname*{argmax}_{k \in \{1..N\}} p_{kt} \longrightarrow \left[\begin{array}{l} \text{in particular,} \\ p_{k_t^*,t} \geqslant 1/N \end{array} \right.$$

Note that k_t^* depends only on ℓ_{js}, $\begin{array}{l} j \in \{1..N\} \\ s \in \{1..t-1\} \end{array}$

The sequence $\begin{cases} \ell_{k_t^*,t} = 1 \\ \ell_{jt} = 0 & j \neq k_t^* \end{cases}$

is such that: $\displaystyle \sum_{t=1}^{T} \sum_j p_{jt}\, \ell_{jt} \geqslant \sum_{t=1}^{T} \underbrace{p_{k_t^*,t}}_{\geqslant 1/N} \ell_{k_t^*,t} \geqslant \frac{T}{N}$

while $\displaystyle \sum_{t=1}^{T} \min_j \ell_{jt} = 0.$

Thus, no strategy can be such that

$$\sup_{\substack{(\ell_{1t} .. \ell_{Nt}) \in \{0,1\}^N \\ t = 1..T}} \left\{ \sum_{t,j} p_{jt}\, \ell_{jt} - \sum_t \min_k \ell_{kt} \right\} = o(T)$$

3) In the proof, instead of applying Hoeffding's lemma

$$\ln \mathbb{E}\big[e^{\eta X}\big] \leq \eta\, \mathbb{E}X + \frac{\eta^2}{8}(M-m)^2$$

we apply Jensen's inequality: $\ln \mathbb{E}\big[e^{\eta X}\big] \geqslant \eta\, \mathbb{E}X$

(valid $\forall \eta \in \mathbb{R}$ and all variables X s.t. X is integrable)

Then $$\sum_j p_{j,t}\, \ell_{j,t} = -\frac{1}{\eta}\underbrace{\Big(-\eta \sum_j p_{j,t}\, \ell_{j,t}\Big)}_{\leqslant \ln \sum_j p_{j,t} e^{-\eta \ell_{j,t}}} \geqslant -\frac{1}{\eta} \ln \sum_j p_{j,t} e^{-\eta \ell_{j,t}}$$

Now, with the same telescoping argument:

$$\sum_{t=1}^{T}\sum_{j=1}^{N} p_{j,t}\, \ell_{j,t} \geqslant -\frac{1}{\eta} \ln \frac{\sum_{j=1}^{N} e^{-\eta \sum_{t=1}^{T} \ell_{j,t}}}{N}$$

$$\geqslant -\frac{1}{\eta} \ln \max_{j=1\ldots N} e^{-\eta \sum_{t=1}^{T} \ell_{j,t}}$$

upper bound the average in the log by the max

$$= \min_{j=1\ldots N} \sum_{t=1}^{T} \ell_{j,t}$$

That, $\forall \eta > 0$, $$R_T = \sum_{t,j} p_{j,t}\, \ell_{j,t} - \min_k \sum_{t=1}^{T} \ell_{k,t} \geqslant 0$$

Exercise #3 / Convex loss functions and comparison to the best convex vector

Strategy at hand: $\eta > 0$ and

$$p_t = \int_{\mathcal{X}} p \, e^{-\eta \sum_{s=1}^{t-1} \ell_s(p)} \, d\mu(p) \Big/ \int_{\mathcal{X}} e^{-\eta \sum_{s=1}^{t-1} \ell_s(p)} \, d\mu(p)$$

$$= \int_{\mathcal{X}} p \, d\mu_t(p) \qquad \text{where} \quad \frac{d\mu_t}{d\mu}(p) = \frac{e^{-\eta \sum_{s=1}^{t-1} \ell_s(p)}}{\int_{\mathcal{X}} e^{-\eta \sum_{s=1}^{t-1} \ell_s(q)} \, d\mu(q)}$$

1)
$$\ell_t(p_t) = \ell_t\Big(\int p \, d\mu_t(p)\Big) \underset{\text{Jensen}}{\leq} \int_{\mathcal{X}} \ell_t(p) \, d\mu_t(p)$$

$$\underset{\text{Hoeffding, as for EWA}}{\leq} -\frac{1}{\eta} \underbrace{\ln \int e^{-\eta \ell_t(p)} d\mu_t(p)}_{\ln \frac{\int e^{-\eta \sum_{s=1}^{t} \ell_s(p)} d\mu(p)}{\int e^{-\eta \sum_{s=1}^{t-1} \ell_s(p)} d\mu(p)}} + \frac{(M-m)^2}{8} \eta$$

Summing over $t = 1, \ldots T$, a telescoping sum appears:

$$\sum_{t=1}^{T} \ell_t(p_t) \leq \underbrace{-\frac{1}{\eta} \ln \frac{\int e^{-\eta \sum_{t=1}^{T} \ell_t(p)} d\mu(p)}{1}}_{\substack{\text{can be bounded using the same} \\ \text{techniques as for} \\ \text{exp. concave losses, but the proof needs to be slightly adapted:}}} + \frac{(M-m)^2}{8} \eta T$$

$\delta > 0$ and p^*_δ s.t. $\inf_{p \in \mathcal{X}} \sum_{t=1}^{T} \ell_t(p) \leq \delta + \sum_{t=1}^{T} \ell_t(p^*_\delta)$

$\varepsilon > 0$ and $\Delta^*_{\delta,\varepsilon} = \{ (1-\varepsilon) p^*_\delta + \varepsilon r, \quad r \in \mathcal{X} \}$

We still have $\mu(\Delta^*_{\delta,\varepsilon}) = \varepsilon^{N-1}$

But for $p = (1-\varepsilon)\, p^*_\delta + \varepsilon r$ we can only resort to convexity:

$$\ell_t(p) \leqslant (1-\varepsilon)\, \ell_t(p^*_\delta) + \varepsilon\, \ell_t(r)$$
$$\leqslant \ell_t(p^*_\delta) + \varepsilon \big(\underbrace{\ell_t(r) - \ell_t(p^*_\delta)}_{\leqslant M-m} \big)$$

since ℓ_t takes values in $[m, M]$ by assumption

$$e^{-\eta \ell_t(p)} \geqslant e^{-\eta \ell_t(p^*_\delta)} \; e^{-\eta \varepsilon (M-m)}$$

Putting all things together:

$$\int_{\mathcal{X}} e^{-\eta \sum_{t=1}^{T} \ell_t(p)} \, d\mu(p) \geqslant e^{-\eta \sum_{t=1}^{T} \ell_t(p^*_\delta)} \times e^{-\eta \varepsilon (M-m) T} \times \varepsilon^{N-1}$$

integral only over $\Delta^*_{\delta, \varepsilon}$

Substituting above and taking $\inf_\varepsilon$:

$$\sum_{t=1}^{T} \ell_t(p_t) \leqslant \underbrace{\sum_{t=1}^{T} \ell_t(p^*_\delta)}_{\leqslant \delta + \inf_p \sum_{t=1}^{T} \ell_t(p)} + \inf_{\varepsilon \in (0,1)} \left\{ \varepsilon (M-m) T - \frac{N-1}{\eta} \ln \varepsilon \right\} + \frac{(M-m)^2}{8} \eta T$$

We let $\delta \downarrow 0$ to conclude:

$$\sum_{t=1}^{T} \ell_t(p_t) - \inf_{p \in \mathcal{X}} \sum_{t=1}^{T} \ell_t(p) \leqslant \frac{(M-m)^2}{8} \eta T + \inf_{\varepsilon \in (0,1)} \left(\varepsilon (M-m) T - \frac{N-1}{\eta} \ln \varepsilon \right)$$

2) Optimize first over ε:

$$g(\varepsilon) = \varepsilon (M-m) T - \frac{N-1}{\eta} \ln \varepsilon$$

g strictly convex,
$$\begin{cases} g'(\varepsilon) = (M-m) T - \dfrac{N-1}{\eta \varepsilon} \\ g''(\varepsilon) = \dfrac{N-1}{\eta \varepsilon^2} > 0 \end{cases}$$

a unique minimizer on $(0, +\infty)$ at ε s.t. $g'(\varepsilon) = 0 \iff \varepsilon = \underbrace{\dfrac{N-1}{\eta (M-m) T}}$

but question is whether this ε is in $(0,1)$!

I tried with this value of ε (which is ok for large T) but couldn't get a simple and readable $O(\sqrt{NT \ln T})$ bound.

Let's not optimize over ε and take an arbitrary choice: $\varepsilon = 1/\sqrt{T}$

The bound is $\leqslant \dfrac{(M-m)^2}{8} \eta T + (M-m)\sqrt{T} + \dfrac{N-1}{2\eta} \ln T$

Optimal value for η: η^* s.t. $\dfrac{(M-m)^2}{8} T \eta^* = \dfrac{N-1}{2\eta^*} \ln T$
(as seen in class)

and for this η^*, the sum is $2 \times \sqrt{\text{the product}}$

$$= \underbrace{\frac{2}{\sqrt{16}}}_{=1/2} (M-m) \sqrt{(N-1) T \ln T}$$

Final bound: $\dfrac{1}{2} (M-m) \sqrt{(N-1) T \ln T} + (M-m)\sqrt{T}$.

$\leadsto$ If you can proceed better, please send me your solution (and you may be rewarded with bonus points at the exam).