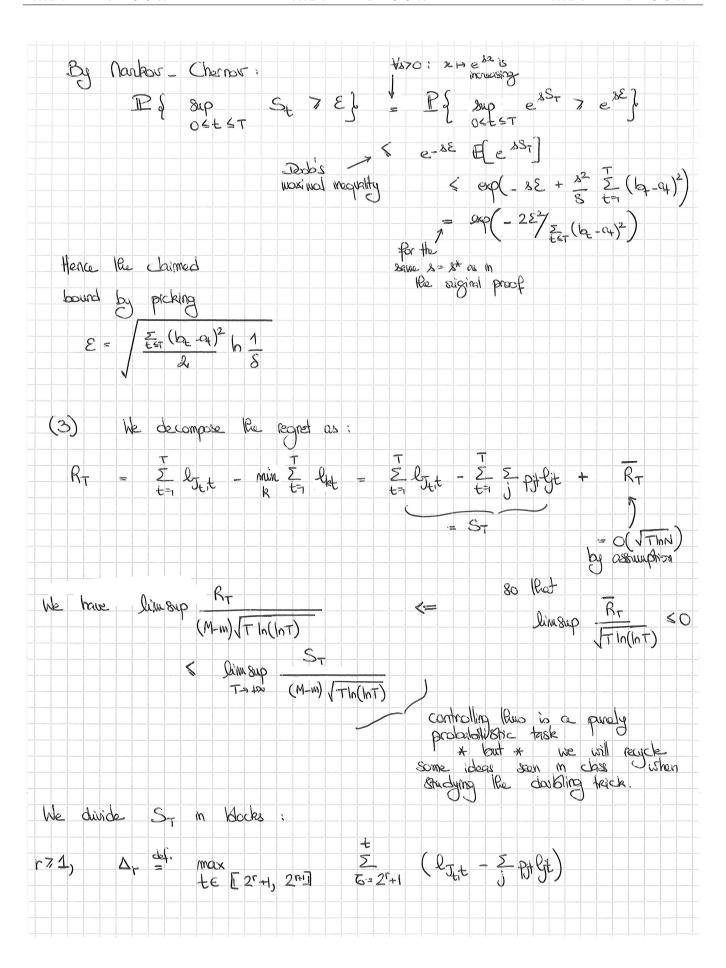
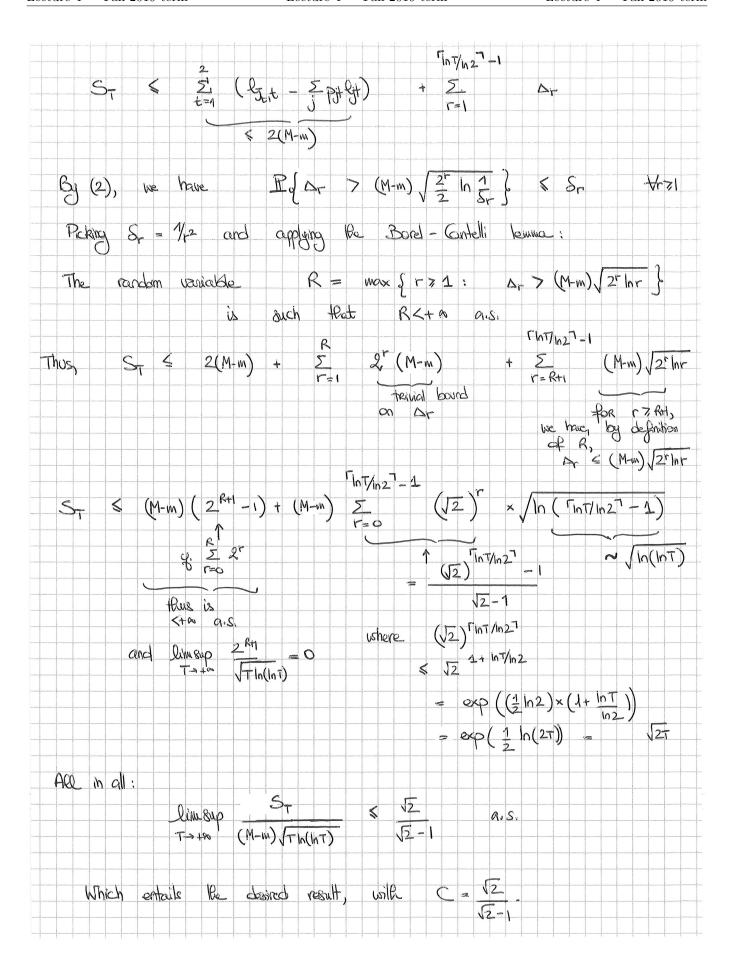


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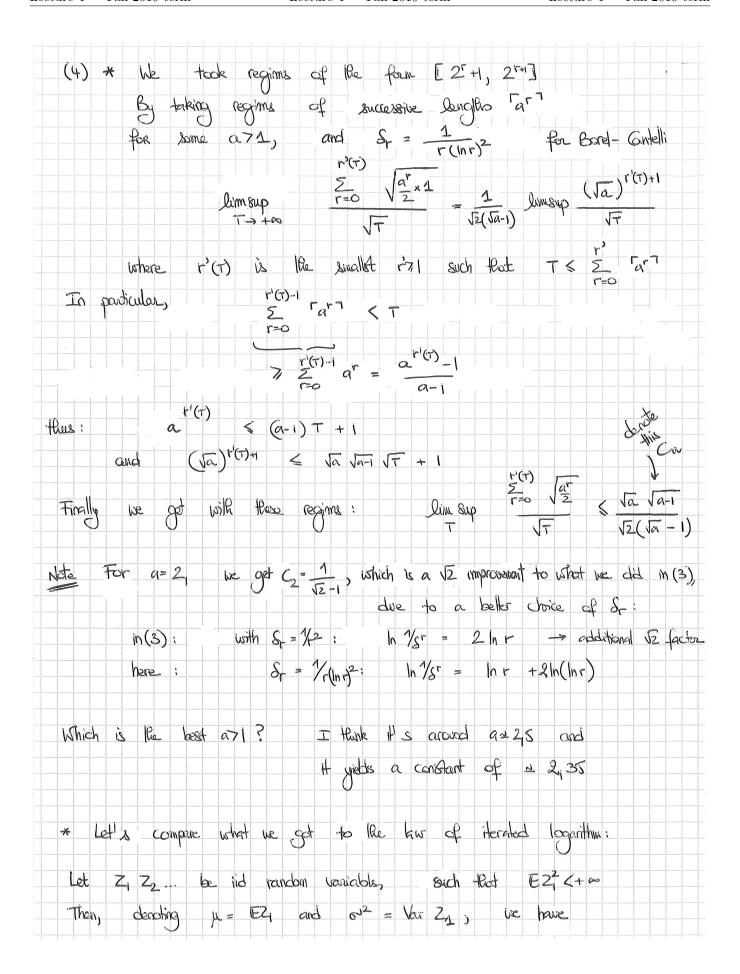


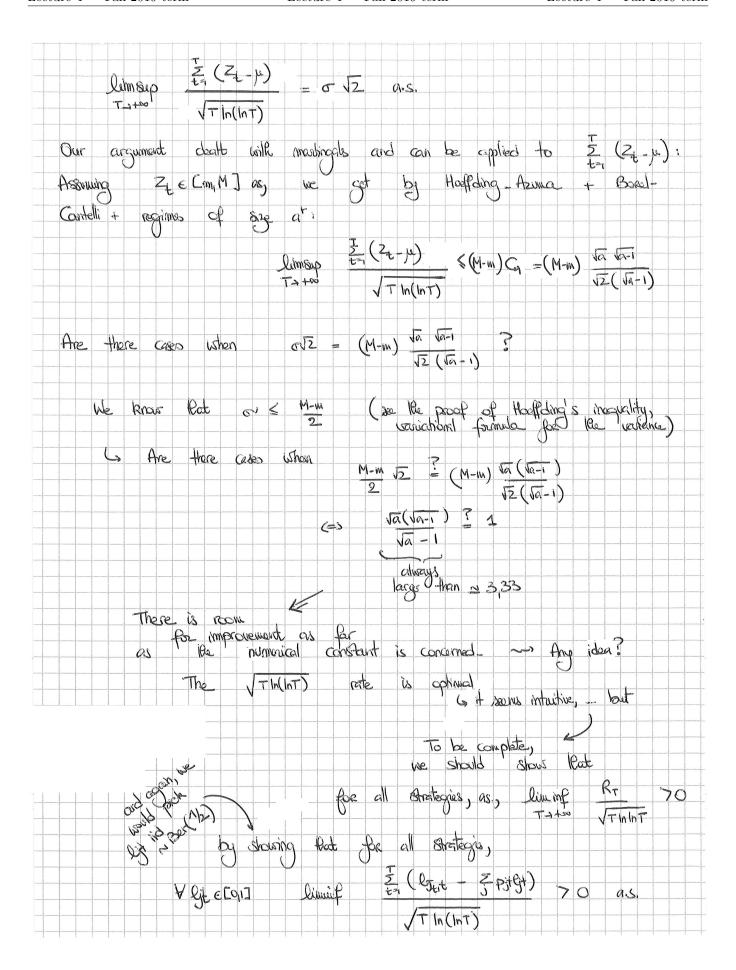
## For question (4) I provide two answers:

- My original answer, where I perform a doubling trick with regimes of lengths given by the integer part of a<sup>r</sup> instead of 2<sup>r</sup>; the constant may be improved but I explain why we still have a gap w.r.t. law of the iterated logarithm
- An answer by Dau Hai Dang (a student who took the course in Spring 2019), where he explains how a modification of the Borel-Cantelli lemma, based on a doubling trick (!), does the job

This all should be some food for thought!

And maybe a clearer summary can be written (also with lower bounds). Please send me your notes if they are worth it!





## Constante optimale pour la borne du regret

12

SPDG, supposons que M-m = 1 et on souhaite Controlor démontrer que

limsup
$$\frac{St}{\sqrt{t \log \log t}} \leq C = : \frac{1}{\sqrt{2}} \quad p.s. \quad (1)$$

où  $S_t = \sum_{n=2}^{t} (l_{J_n,n} - \mathbb{E}[l_{J_n,n}|\mathcal{F}_{n-4}]).$ 

Rappelons que par l'inégalité de Pool, on a

$$\mathbb{P}\left(\sup_{t\leq T}S_{t}\geqslant \varepsilon\right)\leq \exp\left(-\frac{2\varepsilon^{2}}{T}\right). \tag{2}$$

Maintenant, fixons un E >0 et posons Vt l'évènement suivant

$$V_t = \left\{ S_t \leq (C + E) \sqrt{t \log \log t} \right\}.$$

Lemme (Borel-Castelli modifié) Pour démontrer (1), il suffit de démontrer que,

pour tout E>0 et pour un a>1 quelconque, on a

Où BC signifie le complément de l'évenement B.

Preure du lemme (exactement comme la preuve de Borel-Contelli).

L'inégalité signifie que 
$$\mathbb{Z}$$
  $\mathbb{E}\left[\sum_{n}\mathbb{I}_{\left(V_{\lfloor n^{n+1}\rfloor}\cap\cdots\cap V_{\lfloor n^{n+2}\rfloor}\right)^{c}}\right]<+\infty$  donc  $\sum_{n}\mathbb{I}_{\left(V_{\lfloor n^{n+1}\rfloor}\cap\cdots\cap V_{\lfloor n^{n+1}\rfloor}\right)^{c}}<+\infty$  ps

$$\Rightarrow \forall \Lambda > \gamma_0(\omega)$$
:  $\omega \in V_{La^1+1} \cap \dots \cap V_{La^{n+2}}$ 

Retour à la preuve de (1)

$$= \mathbb{P} \Big( \exists t : \lfloor a^r + 1 \rfloor \le t \le \lfloor a^{r+2} \rfloor \ tq \ S_t > (C+\epsilon) \sqrt{t \log \log t} \Big)$$

$$\leq \mathbb{P}\left(\exists t: \lfloor a^{r+2} \rfloor \leq t \leq \lfloor a^{r+2} \rfloor \ \text{tq} \ S_t > (C+\epsilon) \sqrt{a^r \log \log (a^r)}\right)$$

$$\leq P(\exists t: [arta] + 1 \leq t \leq [arta] + q S_t > ((t_E) \sqrt{a^n log log (a^n)})$$

$$\stackrel{(2)}{=} \exp\left(-\frac{2\left((+\xi)^2 a^4 \log \log (a^4)\right)}{\left\lfloor a^{44}\right\rfloor}\right) \leq \exp\left(-\frac{2\left((+\xi)^2 a^4 \log \log (a^4)\right)}{a^{44}}\right)$$

$$\leq \exp\left(-\frac{2((t+\epsilon)^2\log(n\log a)}{a}\right) = \exp\left(-\frac{2((t+\epsilon)^2\log\log a)}{a}\right) e^{-\frac{2((t+\epsilon)^2\log\log a)}{a}}$$

Il suffit danc de choisir a > 1 tel que

$$\sum_{\lambda} r^{-2(C+E)^2} < +\infty.$$

Or, comme 
$$C = \frac{1}{\sqrt{2}}$$
, un tel a existe toujours

Remarque. La constante  $C = \frac{1}{\sqrt{2}}$  est optimale, comme vous avez dit dans le consigé.