(1) $\displaystyle \sup_{\ell_{jt} \in [0,1]} \{ \dots \} \geqslant \mathbb{E}[\quad]$ for any probability distribution over the losses

Thus, $\displaystyle \sup_{\ell_{jt} \in [0,1]} \Big\{ \sum_{t,j} p_{jt} \ell_{jt} - \min_i \sum \ell_{it} \Big\}$

$$(*) \qquad \geqslant \max_{k \in \{1,\dots,N\}} \mathbb{E}_k \Big[\sum_{t,j} p_{jt} L_{jt} - \min_i \sum_{t=1}^{T} L_{it} \Big]$$

Denote by $\mathcal{F}_{t-1} = \sigma\big(L_{js},\ j \in \{1..N\} \text{ and } s \in \{1,\dots,t-1\} \big)$ for $t \geqslant 2$

For $t \geqslant 2$: p_t is $\mathcal{F}_{t-1}$-measurable, so that by the tower rule:

$$\mathbb{E}_k \Big[\sum_{j=1}^{N} p_{jt} L_{jt} \Big] = \mathbb{E}_k \Big[\mathbb{E}\Big[\sum_j p_{jt} L_{jt} \,\Big|\, \mathcal{F}_{t-1} \Big] \Big]$$

$$= \mathbb{E}_k \Big[\sum_j p_{jt} \underbrace{\mathbb{E}_k \big[L_{jt} \,|\, \mathcal{F}_{t-1} \big]}_{\substack{= \mathbb{E}_k[L_{jt}] \text{ by independence of the losses across time} \\ = \begin{cases} 1/2 & \text{if } j \neq k \\ 1/2 - \varepsilon & \text{if } j = k \end{cases}}} \Big]$$

Thus, $$\mathbb{E}_k \Big[\sum_{j=1}^{N} p_{jt} L_{jt} \Big] = \mathbb{E}_k \Big[\sum_{j \neq k} p_{jt}/2 + p_{kt} (1/2 - \varepsilon) \Big]$$

$$= 1/2 - \varepsilon\, \mathbb{E}_k [p_{kt}] \qquad (**)$$

This is also ok for $t = 1$ (in that case, p_1 is constant).

On the other hand, $$\mathbb{E}_k \Big[\min_{i \in \{1..N\}} \sum_{t=1}^{T} L_{it} \Big]$$

$$\leqslant \min_{i \leqslant N} \underbrace{\mathbb{E}_k \Big[\sum_{t=1}^{T} L_{it} \Big]}_{= \begin{cases} T/2 & \text{if } i \neq k \\ T/2 - \varepsilon T & \text{if } i = k \end{cases}} = T/2 - \varepsilon T \qquad (\overset{*}{**})$$

Substituting $(**)$ and $(\overset{*}{**})$ in $(*)$, we get:

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$$\sup_{l_{jt}\in[0,1]} \left\{ \sum_{t,j} p_{jt}\, l_{jt} - \min_{i} \sum_{t=1}^{T} l_{it} \right\}$$

$$\geqslant \max_{k\leq N} \ \mathbb{E}_k\left[\sum_{t,j} p_{jt}\, L_{jt} - \min_{i\leq N} \sum_{t=1}^{T} L_{it} \right]$$

$$\geqslant \max_{k\leq N} \ T\varepsilon \left(1 - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_k\left[p_{kt} \right] \right)$$

don't forget that for $t \geqslant 2$, p_{kt} is a random variable, as it depends on the L_{js}, $j \leq N$ and $s \leq t-1$

<u>IDEA</u>: The idea of the proof is that any strategy will take some time (basically, a time of order $\sqrt{T}$) to identify k as the best arm in $\{1,\dots,N\}$ under $\mathbb{P}_k$.

Since this needs to be performed for N distributions $\mathbb{P}_1,\dots,\mathbb{P}_N$ at a time, an additional $\sqrt{\ln N}$ factor will be gained by Fano's lemma.

(2) Deux choses à voir :

- $\mathrm{kl}(p,q) \leqslant \overline{K}$
- $a \leqslant \mathrm{kl}(p,q)/\ln N$ lorsque $a \geqslant 2e/(2e+1)$

2nd point :

$$\mathrm{kl}(p,q) = p \ln p + (1-p)\ln(1-p) + p \ln \frac{1}{q} + (1-p) \ln\left(\frac{1}{1-q}\right)$$

avec $p = \dfrac{1}{N-1} \displaystyle\sum_{j\geqslant 2} Q_j(A_j) \geqslant a = \min_{j=1,\dots,N} Q_j(A_j)$

et $q = \dfrac{1}{N-1} \displaystyle\sum_{j \geqslant 2} Q_1(A_j) = \dfrac{1}{N-1}\left(1 - Q_1(A_1)\right) \leqslant \dfrac{1}{N-1}(1-a)$

cf. (A_k) est une partition de Ω

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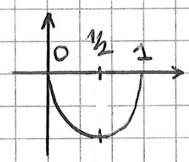
We use that $x \mapsto x \ln x + (1-x) \ln x$ is increasing on $[1/2, 1]$



to get $p \ln p + (1-p) \ln(1-p) \geqslant a \ln a + (1-a) \ln(1-a)$

We use $(1-p) \ln \frac{1}{1-q} \geqslant 0$

We have $p \geqslant a$ and $\frac{1}{q} \geqslant \frac{N-1}{1-a} \geqslant 1$ so that $p \ln \frac{1}{q} \geqslant a \ln\left(\frac{N-1}{1-a}\right)$

All in all, $\mathrm{kl}(p,q) \geqslant a \ln a + (1-a) \ln(1-a) + a \ln\left(\frac{N-1}{1-a}\right)$

$N-1 \geqslant \frac{N}{2}$ $\forall N \geqslant 2$ and $a \ln(N-1) \geqslant a \ln N - a \ln 2$

$$\geqslant a \ln N + \underbrace{\Big[a \ln a + (1-a) \ln(1-a) - a \ln(1-a) - a \ln 2 \Big]}$$

function study: this is $\geqslant 0$ for $a \geqslant 0.76$ (while $2e/(2e+1) \simeq 0.845$)

or (original Birgé's argument)

$$(1-a) \ln(1-a) \geqslant \min_{t \in [0,1]} t \ln t = -1/e \geqslant -2e/(2e+1) \geqslant -a = a \ln \frac{1}{e}$$

thus

$$a \ln a + (1-a) \ln(1-a) - a \ln(1-a) - a \ln 2 \geqslant a \ln \frac{a}{2e(1-a)} \geqslant a \ln\left(\frac{1}{2e} \, \frac{2e/(2e+1)}{1 - 2e/(2e+1)} \right) = 0$$

$t \mapsto \frac{t}{1-t}$ increasing and $a \geqslant 2e/(2e+1)$

In any case:

$$\mathrm{kl}(p,q) \geqslant a \ln N \quad \text{when (eg)} \quad a \geqslant \frac{2e}{2e+1}.$$

1st point $\hookrightarrow$ We now prove that $\mathrm{kl}(p,q) \leqslant \overline{K}$.

The data compression inequality entails that:

- for all distributions μ, ν on $(\Omega, \mathcal{F})$ and any $A \in \mathcal{F}$,

$$\mathrm{kl}\big(\mu(A), \nu(A)\big) \leq \mathrm{KL}(\mu, \nu)$$

indeed, consider $X = \mathbb{1}_A$, then $\mu^X = \mu^{\mathbb{1}_A}$ is the Bernoulli distribution with parameter $\mu(A)$; same for ν; thus:

$$\mathrm{kl}\big(\mu(A), \nu(A)\big) \underset{\text{by definition}}{=} \mathrm{KL}\big(\mu^{\mathbb{1}_A}, \nu^{\mathbb{1}_A}\big) \underset{\text{data-compression inequality}}{\leq} \mathrm{KL}(\mu, \nu)$$

- KL (and thus kl) is jointly convex:

$\forall \alpha \in (0,1)$, $\forall \mu_1, \mu_2$, $\forall \nu_1, \nu_2$,

$$\mathrm{KL}\big(\alpha\mu_1 + (1-\alpha)\mu_2,\ \alpha\nu_1 + (1-\alpha)\nu_2\big) \leq \alpha\, \mathrm{KL}(\mu_1, \nu_1) + (1-\alpha)\, \mathrm{KL}(\mu_2, \nu_2) \qquad (C)$$

other, more direct, proofs exist.

Proof: $\Omega' = \Omega \times \{1,2\}$

$\tilde{\mu}$ on Ω' given by $\forall A \in \mathcal{F}$: $\tilde{\mu}\big(A \times \{j\}\big) = \begin{cases} \alpha\mu_1(A) & \text{if } j=1 \\ (1-\alpha)\mu_2(A) & \text{if } j=2 \end{cases}$

same for $\tilde{\nu}$ based on ν

Let π be the projection $(\omega, j) \in \Omega \times \{1,2\} \longmapsto \omega$

Then $\tilde{\mu}^{\pi} = 1^{\text{st}}$ marginal of $\tilde{\mu}$ $= \alpha\mu_1 + (1-\alpha)\mu_2$

$\tilde{\nu}^{\pi} = \tilde{\nu} = \alpha\nu_1 + (1-\alpha)\nu_2$

The desired inequality holds by data compression:

$$\mathrm{KL}\big(\tilde{\mu}^{\pi}, \tilde{\nu}^{\pi}\big) = \mathrm{KL}\big(\alpha\mu_1 + (1-\alpha)\mu_2,\ \alpha\nu_1 + (1-\alpha)\nu_2\big)$$
$$\leq \mathrm{KL}(\tilde{\mu}, \tilde{\nu}) = ?$$

With no loss of generality we can assume $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$ otherwise the desired inequality (C) is satisfied (its right-hand side $= +\infty$)

Then $\tilde{\mu} \ll \tilde{\nu}$ as well, with $\dfrac{d\tilde{\mu}}{d\tilde{\nu}}(\omega, j) = \dfrac{d\mu_j}{d\nu_j}(\omega)$

$$\mathrm{KL}(\tilde\mu, \tilde\nu) = \int_{\Omega \times \{1,2\}} \left(\ln \frac{d\tilde\mu}{d\tilde\nu} \right) d\tilde\nu$$

$$= \alpha \times \int_\Omega \left(\ln \frac{d\mu_1}{d\tilde\nu_1} \right) d\tilde\nu_1 + (1-\alpha) \int_\Omega \left(\ln \frac{d\mu_2}{d\tilde\nu_2} \right) d\tilde\nu_2$$

$$= \alpha \, \mathrm{KL}(\mu_1, \tilde\nu_1) + (1-\alpha) \, \mathrm{KL}(\mu_2, \tilde\nu_2).$$

Application:

$$\mathrm{kl}(p,q) = \mathrm{kl}\left(\frac{1}{N-1} \sum_{j \geq 2} \mathbb{Q}_j(A_j), \frac{1}{N-1} \sum_{j \geq 2} \mathbb{Q}_1(A_j) \right)$$

$$\underset{\text{joint convexity of kl}}{\leqslant} \frac{1}{N-1} \sum_{j \geq 2} \mathrm{kl}\big(\mathbb{Q}_j(A_j), \mathbb{Q}_1(A_j) \big)$$

$$\underset{\text{data-compression ineq.}}{\leqslant} \frac{1}{N-1} \sum_{j \geq 2} \mathrm{KL}(\mathbb{Q}_j, \mathbb{Q}_1) = \overline{K}.$$

(3) We denote $a_k = \mathbb{E}_k\left[\frac{1}{T} \sum_{t=1}^T p_{kt} \right]$ and $b_k = \mathbb{E}_1\left[\frac{1}{T} \sum_{t=1}^T p_{kt} \right]$ $(k \geq 1)$

note that $b_1 = a_1$

The "2nd part" of the proof in (2) was purely analytical and only used that $b_1 + b_2 + \cdots + b_N = 1$, which is still true.

Therefore, we similarly get:

$$\min_{k=1,\ldots,N} \mathbb{E}_k\left[\frac{1}{T} \sum_{t=1}^T p_{kt} \right] \leqslant \max\left\{ \frac{2e}{2e+1}, \; \mathrm{kl}\left(\frac{1}{N-1} \sum_{j \geq 2} \mathbb{E}_j\left[\frac{1}{T} \sum_t p_{jt} \right], \frac{1}{N-1} \sum_{j \geq 2} \mathbb{E}_1\left[\frac{1}{T} \sum_t p_{jt} \right] \right) \times 1/\ln N \right\}$$

By convexity of kl, we may further upper bound the right-hand side by

$$\max\left\{ \frac{2e}{2e+1}, \frac{1}{\ln N} \left(\frac{1}{N-1} \sum_{j=2}^N \mathrm{kl}\left(\mathbb{E}_j\left[\frac{1}{T} \sum_t p_{jt} \right], \mathbb{E}_1\left[\frac{1}{T} \sum_t p_{jt} \right] \right) \right) \right\}$$

It thus suffices to show that $\mathrm{kl}(\mathbb{E}_j Z, \mathbb{E}_1 Z) \leqslant \mathrm{KL}(\mathbb{P}_j^L, \mathbb{P}_1^L)$ for any random variable Z that

- takes values in $[0,1]$
- is $\sigma(L)$-measurable

} that is,

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random variables Z of the form $Z = \Psi(L)$

where $\Psi : \ell = (\ell_{j,t})_{\substack{j \le N \\ t \le T}} \longmapsto \Psi(\ell) \in [0,1]$ is measurable.

But $\mathbb{E}_j Z = \mathbb{E}_j \Psi(L)$

$$= \int \Psi(\ell)\, d\mathbb{P}_j^L(\ell)$$

where $\mathbb{P}_j^L$ is the image distribution of $\mathbb{P}_j$ by L.

The result thus follows from the two reminders of the properties of the KL divergence (see first page of the statement of the exercise):

Lemma: Let $f : (\Omega, \mathcal{F}) \to [0,1]$ be measurable and let μ, ν be probability distributions over $(\Omega, \mathcal{F})$. Then:

$$\mathrm{kl}\left(\int_\Omega f\, d\mu,\ \int_\Omega f\, d\nu \right) \le \mathrm{KL}(\mu, \nu).$$

Proof: Let $\tilde{\Omega} = \Omega \times [0,1]$ (equipped with the product σ-algebra)

Let $E = \{ (\omega, t) \text{ s.t. } f(\omega) \ge t \}$; E is measurable

Let $\tilde{\mu} = \mu \otimes \lambda$ and $\tilde{\nu} = \nu \otimes \lambda$ where λ is the Lebesgue measure on $[0,1]$

$$\mathrm{KL}(\mu, \nu) = \mathrm{KL}(\tilde{\mu}, \tilde{\nu}) \ge \mathrm{kl}\big(\tilde{\mu}(E), \tilde{\nu}(E) \big)$$

cf. 1st reminder on KL divergence as $\mathrm{KL}(\lambda, \lambda) = 0$

a special case of data compression with $X = \mathbb{1}_E$

But by Fubini-Tonelli: $\tilde{\mu}(E) = \displaystyle\iint \mathbb{1}_{\{f(\omega) \ge t\}}\, d\mu(\omega)\, dt$ integrating over t

$$= \int f(\omega)\, d\mu(\omega)$$

and similarly for $\tilde{\nu}(E)$.

(4) By independence, $\mathbb{P}_j^L$ is a product of NT distributions. Using that

$$KL(\mu \otimes \mu', \nu \otimes \nu') = KL(\mu, \nu) + KL(\mu', \nu')$$

(iterating this equality), we get

$$KL(\mathbb{P}_j^L, \mathbb{P}_1^L) = \sum_{k,t} \underbrace{KL(\mathbb{P}_j^{L_{kt}}, \mathbb{P}_1^{L_{kt}})}$$

$$\begin{cases} = 0 & \text{if } k \neq 1 \text{ and } k \neq j \\ = KL(\mathrm{Ber}(1/2-\varepsilon), \mathrm{Ber}(1/2)) & \text{if } k = j \\ = KL(\mathrm{Ber}(1/2), \mathrm{Ber}(1/2-\varepsilon)) & \text{if } k = 1 \end{cases}$$

Thus, $\forall j$, $$KL(\mathbb{P}_j^L, \mathbb{P}_1^L) = T \times \Big(\underbrace{kl(1/2-\varepsilon, 1/2) + kl(1/2, 1/2-\varepsilon)}_{\text{it suffices to show that this is } \leqslant 5\varepsilon^2 \text{ when } \varepsilon \leqslant 1/10.} \Big)$$

$$kl(1/2-\varepsilon, 1/2) + kl(1/2, 1/2-\varepsilon)$$
$$= \left(\frac{1}{2}-\varepsilon\right) \ln \frac{1/2-\varepsilon}{1/2} + \big(1-(1/2-\varepsilon)\big) \ln \frac{1-(1/2-\varepsilon)}{1/2} + \frac{1}{2} \ln \frac{1/2}{1/2-\varepsilon} + \frac{1}{2} \ln \frac{1/2}{1-(1/2-\varepsilon)}$$
$$= (1/2-\varepsilon) \ln(1-2\varepsilon) + (1/2+\varepsilon) \ln(1+2\varepsilon) - \frac{1}{2} \ln(1-2\varepsilon) - \frac{1}{2} \ln(1+2\varepsilon)$$
$$= \varepsilon \ln(1+2\varepsilon) - \varepsilon \ln(1-2\varepsilon) = \varepsilon \ln\big((1+2\varepsilon)/(1-2\varepsilon) \big)$$
$$= \varepsilon \ln\left(1 + \frac{4\varepsilon}{1-2\varepsilon}\right) \underset{\ln(1+u) \leqslant u}{\leqslant} \frac{4\varepsilon^2}{1-2\varepsilon} \underset{\substack{\text{for } \varepsilon \leqslant 1/10, \\ 1-2\varepsilon \geqslant 4/5}}{\leqslant} 5\varepsilon^2$$

Hence $$\overline{K}' = \frac{1}{N-1} \sum_{j \geqslant 2} KL(\mathbb{P}_j^L, \mathbb{P}_1^L)$$
$$= T\big(kl(1/2-\varepsilon, 1/2) + kl(1/2, 1/2-\varepsilon) \big) \leqslant 5T\varepsilon^2 \quad \text{for } \varepsilon \leqslant 1/10.$$

(5) Questions (1) - (4) lead to $\forall \varepsilon \in (0, 1/10]$,

$$SR_T \overset{\text{def}}{=} \sup_{\ell_{jt} \in [0,1]} \left\{ \sum_{t,j} p_{jt} \ell_{jt} - \min_k \sum_{t=1}^{T} \ell_{kt} \right\} \geqslant T\varepsilon \left(1 - \min_k \mathbb{E}_k\left[\frac{1}{T} \sum_t p_{kt} \right] \right)$$
$$\geqslant T\varepsilon \left(1 - \max\left\{ \frac{2e}{2e+1}, \frac{5T\varepsilon^2}{\ln N} \right\} \right)$$

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We would like to take, e.g., ε such that $5T\varepsilon^2/\ln N = \frac{2e}{2e+1}$

that is, $\varepsilon^* = \sqrt{\frac{2e}{2e+1}\,\frac{\ln N}{5T}}$

This ε^* is $\leq 1/10$ when

$$\frac{\ln N}{T} \leq \frac{(2e+1)5}{2e} \times \frac{1}{100} \approx 0,059197$$

and $1/17 \approx 0,58823$

Thus, $\varepsilon^* \leq 1/10$ when $T \geq 17 \ln N$.

With this ε^*, the bound becomes

$$T\,\varepsilon^*\left(1 - \frac{2e}{2e+1}\right)$$

$$= \sqrt{T\ln N} \times \Bigg(\underbrace{\sqrt{\frac{2e}{(2e+1)5}} \times \frac{1}{2e+1}}_{\geq 0.06} \Bigg)$$

<u>Theorem</u> For all strategies, for all $N \geq 2$, for all $T \geq 17 \ln N$,

$$\sup_{\ell_{jt}\in[0,1]} \left\{ \sum_{t,j} p_{jt}\,\ell_{jt} - \min_k \sum_t \ell_{kt} \right\} \geq 0.06\sqrt{T\ln N}$$

PS There will be bonus points for those who will significantly improve both constants 17 and 0.06 ! In particular, the 0.06 should become as close as possible to $1/\sqrt{2} \simeq 0,7$.