 Lecture 1 — Spring 2021 term Lecture 1 — Spring 2021 term

## Exercise #1 / Various considerations around the notion of regret.

1) ($N = 2$ is enough)

| Time | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ | $t=6$ | etc. |
|---|---|---|---|---|---|---|---|
| $\ell_{1t}$ | 1 | 0 | 1 | 0 | 1 | 0 | |
| $\sum_{s=1}^{t-1} \ell_{1s}$ | | 1 | 1 | 2 | 2 | 3 | |
| $\ell_{2t}$ | $1/2$ | 1 | 0 | 1 | 0 | 1 | |
| $\sum_{s=1}^{t-1} \ell_{2s}$ | | $1/2$ | $1+1/2$ | $1+1/2$ | $2+1/2$ | $2+1/2$ | |
| Leader over $s=1,\ldots,t-1$ | | 2 | 1 | 2 | 1 | 2 | |
| $p_{1t}$ | $1/2$ | 0 | 1 | 0 | 1 | 0 | |
| $p_{2t}$ | $1/2$ | 1 | 0 | 1 | 0 | 1 | |
| $\sum_{j=1}^{2} p_{jt}\ell_{jt}$ | $\frac{3}{4}$ | 1 | 1 | 1 | 1 | 1 | |
| $\sum_{s=1}^{t} \sum_j p_{jt}\ell_{jt}$ | $\frac{3}{4}$ | $1+\frac{3}{4}$ | $2+\frac{3}{4}$ | $3+\frac{3}{4}$ | $4+\frac{3}{4}$ | $5+\frac{3}{4}$ | |

Follow the leader is a strategy that fails miserably $\longrightarrow$ EWA is a smoothed alternative.

We have
$$\sum_{t=1}^{T} \sum_{j=1}^{2} p_{jt}\ell_{jt} = T-1+\frac{3}{4}$$

$$\sum_{t=1}^{T} \ell_{1t} = \begin{cases} T/2 & \text{if } T \text{ even} \\ \frac{T-1}{2} & \text{if } T \text{ odd} \end{cases} \qquad \text{(lower integer part)}$$

$$\sum_{t=1}^{T} \ell_{2t} = \begin{cases} (T-1)/2 + 1/2 & \text{if } T \text{ odd} \\ T/2 - 1 + 1/2 & \text{if } T \text{ even} \end{cases}$$

$$\min\left\{ \sum_{t=1}^{T} \ell_{1t},\ \sum_{t=1}^{T} \ell_{2t} \right\} \leqslant \frac{T}{2}$$

$$R_T \geqslant T-1+\frac{3}{4} - T/2 = T/2 - 1/4 \qquad \neq o(T)$$

2) It suffices to consider only the sequences in $\{0,1\}^{NT}$ ;

For any given strategy, we denote for $t \geqslant 2$ :

$$k_t^* \in \operatorname*{argmax}_{k \in \{1..N\}} p_{kt} \longrightarrow \left[\begin{array}{l}\text{in particular,}\\ p_{k_t^*,t} \geqslant 1/N\end{array}\right.$$

Note that $k_t^*$ depends only on $\ell_{js}$, $\begin{array}{l} j \in \{1..N\} \\ s \in \{1..t-1\}\end{array}$

The sequence $\left\{\begin{array}{ll}\ell_{k_t^*,t} = 1 & \\ \ell_{jt} = 0 & j \neq k_t^*\end{array}\right.$

is such that : $\displaystyle \sum_{t=1}^{T} \sum_j p_{jt}\, \ell_{jt} \geqslant \sum_{t=1}^{T} \underbrace{p_{k_t^*,t}}_{\geqslant 1/N} \ell_{k_t^*,t} \geqslant \frac{T}{N}$

while $\displaystyle \sum_{t=1}^{T} \min_j \ell_{jt} = 0.$

Thus, <u>no</u> strategy can be such that

$$\sup_{\substack{(\ell_{1t} .. \ell_{Nt}) \in \{0,1\}^N \\ t=1..T}} \left\{ \sum_{t,j} p_{jt}\, \ell_{jt} - \sum_t \min_k \ell_{kt} \right\} = o(T)$$

3) In the proof, instead of applying Hoeffding's lemma

$$\ln \mathbb{E}[e^{\eta X}] \leq \eta \mathbb{E}X + \frac{\eta^2}{8}(M-m)^2$$

we apply Jensen's inequality: $\ln \mathbb{E}[e^{\eta X}] \geqslant \eta \mathbb{E}X$

(valid $\forall \eta \in \mathbb{R}$ and all variables $X$ s.t. $X$ is integrable)

Then
$$\sum_j p_{jt} \ell_{jt} = -\frac{1}{\eta}\Big(\underbrace{-\eta \sum_j p_{jt}\ell_{jt}}_{\leqslant \ln \sum_j p_{jt} e^{-\eta \ell_{jt}}}\Big) \geqslant -\frac{1}{\eta} \ln \sum_j p_{jt} e^{-\eta \ell_{jt}}$$

Now, with the same telescoping argument:

$$\sum_{t=1}^T \sum_{j=1}^N p_{jt} \ell_{jt} \geqslant -\frac{1}{\eta} \ln \frac{\sum_{j=1}^N e^{-\eta \sum_{t=1}^T \ell_{jt}}}{N}$$

upper bound the average by the max in the log

$$\geqslant -\frac{1}{\eta} \ln \max_{j=1..N} e^{-\eta \sum_{t=1}^T \ell_{jt}}$$

$$= \min_{j=1..N} \sum_{t=1}^T \ell_{jt}$$

That, $\forall \eta > 0$, $\quad R_T = \sum_{t,j} p_{jt}\ell_{jt} - \min_k \sum_{t=1}^T \ell_{kt} \geqslant 0$

Exercise #3 / Convex loss functions and comparison to the best convex vector

Strategy at hand: $\eta > 0$ and

$$p_t = \int_{\mathcal{X}} p \, e^{-\eta \sum_{s=1}^{t-1} \ell_s(p)} \, d\mu(p) \Big/ \int_{\mathcal{X}} e^{-\eta \sum_{s=1}^{t-1} \ell_s(p)} \, d\mu(p)$$

$$= \int_{\mathcal{X}} p \, d\mu_t(p) \qquad \text{where} \quad \frac{d\mu_t}{d\mu}(p) = \frac{e^{-\eta \sum_{s=1}^{t-1} \ell_s(p)}}{\int_{\mathcal{X}} e^{-\eta \sum_{s=1}^{t-1} \ell_s(q)} \, d\mu(q)}$$

1) $$\ell_t(p_t) = \ell_t\Big(\int p \, d\mu_t(p)\Big) \underset{\text{Jensen}}{\leqslant} \int_{\mathcal{X}} \ell_t(p) \, d\mu_t(p)$$

$$\underset{\text{Hoeffding, as for EWA}}{\leqslant} -\frac{1}{\eta} \underbrace{\ln \int e^{-\eta \ell_t(p)} d\mu_t(p)}_{\ln \frac{\int e^{-\eta \sum_{s=1}^{t} \ell_s(p)} d\mu(p)}{\int e^{-\eta \sum_{s=1}^{t-1} \ell_s(p)} d\mu(p)}} + \frac{(M-m)^2}{8} \eta$$

Summing over $t = 1, \ldots T$, a telescoping sum appears:

$$\sum_{t=1}^{T} \ell_t(p_t) \leqslant \underbrace{-\frac{1}{\eta} \ln \frac{\int e^{-\eta \sum_{t=1}^{T} \ell_t(p)} d\mu(p)}{1}}_{\text{can be bounded using the same techniques as for exp. concave losses, but the proof needs to be slightly adapted:}} + \frac{(M-m)^2}{8} \eta T$$

$\delta > 0$ and $p^*_\delta$ s.t. $\displaystyle \inf_{p \in \mathcal{X}} \sum_{t=1}^{T} \ell_t(p) \leqslant \delta + \sum_{t=1}^{T} \ell_t(p^*_\delta)$

$\varepsilon > 0$ and $\Delta^*_{\delta,\varepsilon} = \{ (1-\varepsilon) p^*_\delta + \varepsilon r, \quad r \in \mathcal{X} \}$

We still have $\mu(\Delta^*_{\delta,\varepsilon}) = \varepsilon^{N-1}$

But for $p = (1-\varepsilon)\, p^*_\delta + \varepsilon r$ we can only resort to convexity:

$$\ell_t(p) \leqslant (1-\varepsilon)\, \ell_t(p^*_\delta) + \varepsilon\, \ell_t(r)$$
$$\leqslant \ell_t(p^*_\delta) + \varepsilon \big( \underbrace{\ell_t(r) - \ell_t(p^*_\delta)}_{\leqslant M-m \text{ since } \ell_t \text{ takes values in } [m,M] \text{ by assumption}} \big)$$

$$e^{-\eta \ell_t(p)} \geqslant e^{-\eta \ell_t(p^*_\delta)} \, e^{-\eta \varepsilon (M-m)}$$

Putting all things together:

$$\int_{\mathcal{X}} e^{-\eta \sum_{t=1}^T \ell_t(p)} \, d\mu(p) \geqslant e^{-\eta \sum_{t=1}^T \ell_t(p^*_\delta)} \times e^{-\eta \varepsilon (M-m) T} \times \varepsilon^{N-1}$$

integral only over $\Delta^*_{\delta,\varepsilon}$

Substituting above and taking $\inf_\varepsilon$:

$$\sum_{t=1}^T \ell_t(p_t) \leqslant \underbrace{\sum_{t=1}^T \ell_t(p^*_\delta)}_{\leqslant \delta + \inf_p \sum_{t=1}^T \ell_t(p)} + \inf_{\varepsilon \in (0,1)} \left\{ \varepsilon (M-m) T - \frac{N-1}{\eta} \ln \varepsilon \right\} + \frac{(M-m)^2}{8} \eta T$$

We let $\delta \downarrow 0$ to conclude:

$$\sum_{t=1}^T \ell_t(p_t) - \inf_{p \in \mathcal{X}} \sum_{t=1}^T \ell_t(p) \leqslant \frac{(M-m)^2}{8} \eta T + \inf_{\varepsilon \in (0,1)} \left( \varepsilon (M-m) T - \frac{N-1}{\eta} \ln \varepsilon \right)$$

2) Optimize first over $\varepsilon$:

$$g(\varepsilon) = \varepsilon (M-m) T - \frac{N-1}{\eta} \ln \varepsilon$$

$g$ strictly convex, a unique minimizer on $(0,+\infty)$ at $\varepsilon$ s.t. $g'(\varepsilon) = 0$

$$\left\{ \begin{array}{l} g'(\varepsilon) = (M-m) T - \dfrac{N-1}{\eta \varepsilon} \\ g''(\varepsilon) = \dfrac{N-1}{\eta \varepsilon^2} > 0 \end{array} \right.$$

$$g'(\varepsilon) = 0 \iff \underbrace{\varepsilon = \frac{N-1}{\eta (M-m) T}}$$

but question is whether this $\varepsilon$ is in $(0,1)$!

I tried with this value of $\varepsilon$ (which is ok for large T) but couldn't get a simple and readable $O(\sqrt{NT\ln T})$ bound.

Let's not optimize over $\varepsilon$ and take an arbitrary choice: $\varepsilon = 1/\sqrt{T}$

The bound is $\leqslant \dfrac{(M-m)^2}{8}\eta T + (M-m)\sqrt{T} + \dfrac{N-1}{2\eta}\ln T$

Optimal value for $\eta$: $\eta^*$ s.t. $\dfrac{(M-m)^2}{8} T\eta^* = \dfrac{N-1}{2\eta^*}\ln T$ (as seen in class)

and for this $\eta^*$, the sum is $2\times\sqrt{\text{the product}}$

$$= \underbrace{\frac{2}{\sqrt{16}}}_{=1/2} (M-m)\sqrt{(N-1)T\ln T}$$

Final bound: $\dfrac{1}{2}(M-m)\sqrt{(N-1)T\ln T} + (M-m)\sqrt{T}$.

$\rightsquigarrow$ If you can proceed better, please send me your solution (and you may be rewarded with bonus points at the exam).