(1) $\sup_{\ell_{jt}\in[0,1]} \{ \dots \} \geqslant \mathbb{E}[\quad]$ for any probability distribution over the losses

Thus, $\sup_{\ell_{jt}\in[0,1]} \left\{ \sum_{t,j} p_{jt}\ell_{jt} - \min_i \sum \ell_{it} \right\}$

(*) $\geqslant \max_{k\in\{1,\dots,N\}} \mathbb{E}_k\left[\sum_{t,j} p_{jt} L_{jt} - \min_i \sum_{t=1}^{T} L_{it} \right]$

Denote by $\mathcal{F}_{t-1} = \sigma\left(L_{js},\ j\in\{1..N\} \text{ and } s\in\{1,\dots,t-1\}\right)$ for $t\geqslant 2$

For $t\geqslant 2$: p_t is $\mathcal{F}_{t-1}$-measurable, so that by the tower rule:

$$\mathbb{E}_k\left[\sum_{j=1}^{N} p_{jt} L_{jt}\right] = \mathbb{E}_k\left[\mathbb{E}\left[\sum_j p_{jt} L_{jt} \,\middle|\, \mathcal{F}_{t-1}\right]\right]$$

$$= \mathbb{E}_k\left[\sum_j p_{jt}\, \underbrace{\mathbb{E}_k\left[L_{jt} \mid \mathcal{F}_{t-1}\right]}_{\substack{= \mathbb{E}_k[L_{jt}] \text{ by independence of the losses across time} \\ = \begin{cases} 1/2 & \text{if } j\neq k \\ 1/2-\varepsilon & \text{if } j=k \end{cases}}}\right]$$

Thus, $\mathbb{E}_k\left[\sum_{j=1}^{N} p_{jt} L_{jt}\right] = \mathbb{E}_k\left[\sum_{j\neq k} p_{jt}/2 + p_{kt}(1/2-\varepsilon)\right]$

$$= 1/2 - \varepsilon\, \mathbb{E}_k[p_{kt}] \qquad (**)$$

This is also ok for $t=1$ (in that case, p_1 is constant).

On the other hand, $\mathbb{E}_k\left[\min_{i\in\{1..N\}} \sum_{t=1}^{T} L_{it}\right]$

$$\leqslant \min_{i\leq N} \underbrace{\mathbb{E}_k\left[\sum_{t=1}^{T} L_{it}\right]}_{= \begin{cases} T/2 & \text{if } i\neq k \\ T/2-\varepsilon T & \text{if } i=k \end{cases}} = T/2 - \varepsilon T \qquad (\overset{*}{**})$$

Substituting $(**)$ and $(\overset{*}{**})$ in $(*)$, we get:

$$\sup_{l_{jt}\in[0,1]} \left\{ \sum_{t,j} p_{jt} l_{jt} - \min_i \sum_{t=1}^T l_{it} \right\}$$

$$\geq \max_{k\leq N} \mathbb{E}_k \left[\sum_{t,j} p_{jt} L_{jt} - \min_{i\leq N} \sum_{t=1}^T L_{it} \right]$$

$$\geq \max_{k\leq N} T\varepsilon \left(1 - \frac{1}{T} \sum_{t=1}^T \mathbb{E}_k [p_{kt}] \right)$$

don't forget that for $t \geq 2$, p_{kt} is a random variable, as it depends on the L_{js}, $j\leq N$ and $s \leq t-1$

IDEA: The idea of the proof is that any strategy will take some time (basically, a time of order $\sqrt{T}$) to identify k as the best arm in $\{1,\dots N\}$ under $\mathbb{P}_k$.

Since this needs to be performed for N distributions $\mathbb{P}_1, \dots \mathbb{P}_N$ at a time, an additional $\sqrt{\ln N}$ factor will be gained by Fano's lemma.

(2) Deux choses à voir :

- $kl(p,q) \leq \overline{K}$
- $a \leq kl(p,q)/\ln N$ lorsque $a \geq 2e/(2e+1)$

2nd point : $$kl(p,q) = p \ln p + (1-p) \ln (1-p) + p \ln \frac{1}{q} + (1-p) \ln \left(\frac{1}{1-q} \right)$$

avec $$p = \frac{1}{N-1} \sum_{j\geq 2} Q_j(A_j) \geq a = \min_{j=1\dots N} Q_j(A_j)$$

et $$q = \frac{1}{N-1} \sum_{j\geq 2} Q_1(A_j) = \frac{1}{N-1} \left(1 - Q_1(A_1) \right) \leq \frac{1}{N-1} (1-a)$$

cf. (A_k) est une partition de Ω

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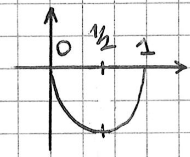
We use that $x \mapsto x\ln x + (1-x)\ln x$ is increasing on $[1/2, 1]$



to get $p\ln p + (1-p)\ln(1-p) \geqslant a\ln a + (1-a)\ln(1-a)$

We use $(1-p)\ln\frac{1}{1-q} \geqslant 0$

We have $p \geqslant a$ and $\frac{1}{q} \geqslant \frac{N-1}{1-a} \geqslant 1$ so that $p\ln\frac{1}{q}$

$\geqslant a \ln\left(\frac{N-1}{1-a}\right)$

All in all, $kl(p,q) \geqslant a\ln a + (1-a)\ln(1-a) + a\ln\left(\frac{N-1}{1-a}\right)$

$N-1 \geqslant \frac{N}{2} \quad \forall N \geqslant 2$

and $a\ln(N-1) \geqslant a\ln N - a\ln 2$

$$\geqslant a\ln N + \underbrace{\Big[a\ln a + (1-a)\ln(1-a) - a\ln(1-a) - a\ln 2 \Big]}$$

function study: this is $\geqslant 0$ for $a \geqslant 0.76$
(while $2e/(2e+1) \simeq 0.845$)

or (original Birgé's argument)

$(1-a)\ln(1-a) \geqslant \min_{t\in[0,1]} t\ln t = -1/e$

$\geqslant -2e/(2e+1) \geqslant -a$

$= a\ln\frac{1}{e}$

thus

$a\ln a + (1-a)\ln(1-a)$
$- a\ln(1-a) - a\ln 2$
$\geqslant a\ln\frac{a}{2e(1-a)} \geqslant a\ln\left(\frac{1}{2e}\,\frac{2e/(2e+1)}{1-2e/(2e+1)}\right) = 0$

$\uparrow$

$t \mapsto \frac{t}{1-t}$ increasing and $a \geqslant 2e/2e+1$

In any case:

$$kl(p,q) \geqslant a\ln N \quad \text{when (eg)} \quad a \geqslant \frac{2e}{2e+1}.$$

1st point $\hookrightarrow$ We now prove that $kl(p,q) \leqslant \overline{K}$.

The data compression inequality entails that:

- for all distributions μ, ν on $(\Omega, \mathcal{F})$ and any $A \in \mathcal{F}$,

$$\mathrm{kl}\big(\mu(A), \nu(A)\big) \leqslant \mathrm{KL}(\mu, \nu)$$

indeed, consider $X = \mathbb{1}_A$, then $\mu^X = \mu^{\mathbb{1}_A}$ is the Bernoulli distribution with parameter $\mu(A)$; same for ν; thus:

$$\mathrm{kl}\big(\mu(A), \nu(A)\big) \underset{\text{by definition}}{=} \mathrm{KL}\big(\mu^{\mathbb{1}_A}, \nu^{\mathbb{1}_A}\big) \underset{\substack{\text{data-}\\ \text{compression inequality.}}}{\leqslant} \mathrm{KL}(\mu, \nu)$$

- KL (and thus kl) is jointly convex:

$\forall \alpha \in (0,1)$, $\forall \mu_1, \mu_2$, $\forall \nu_1, \nu_2$,

$$\mathrm{KL}\big(\alpha\mu_1 + (1-\alpha)\mu_2,\ \alpha\nu_1 + (1-\alpha)\nu_2\big) \leqslant \alpha\,\mathrm{KL}(\mu_1, \nu_1) + (1-\alpha)\,\mathrm{KL}(\mu_2, \nu_2) \qquad (C)$$

other, more direct, proofs exist.

Proof: $\Omega' = \Omega \times \{1,2\}$

$\tilde{\mu}$ on Ω' given by $\forall A \in \mathcal{F}$: $\tilde{\mu}\big(A \times \{j\}\big) = \begin{cases} \alpha\,\mu_1(A) & \text{if } j=1 \\ (1-\alpha)\mu_2(A) & \text{if } j=2 \end{cases}$

same for $\tilde{\nu}$ based on ν

Let π be the projection $(\omega, j) \in \Omega \times \{1,2\} \longmapsto \omega$

Then $\tilde{\mu}^{\pi} = 1^{\text{st}}$ marginal of $\tilde{\mu} = \alpha\mu_1 + (1-\alpha)\mu_2$

$\tilde{\nu}^{\pi} = \phantom{1^{\text{st}} \text{ marginal of }} \tilde{\nu} = \alpha\nu_1 + (1-\alpha)\nu_2$

The desired inequality holds by data compression:

$$\mathrm{KL}\big(\tilde{\mu}^{\pi}, \tilde{\nu}^{\pi}\big) = \mathrm{KL}\big(\alpha\mu_1 + (1-\alpha)\mu_2,\ \alpha\nu_1 + (1-\alpha)\nu_2\big)$$

$$\leqslant \mathrm{KL}(\tilde{\mu}, \tilde{\nu}) = ?$$

With no loss of generality we can assume $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$ otherwise the desired inequality (C) is satisfied (its right-hand side $= +\infty$)

Then $\tilde{\mu} \ll \tilde{\nu}$ as well, with $\dfrac{d\tilde{\mu}}{d\tilde{\nu}}(\omega, j) = \dfrac{d\mu_j}{d\nu_j}(\omega)$

$$\mathrm{KL}(\tilde{\mu},\tilde{\nu}) = \int_{\Omega\times\{1,2\}} \left(\ln \frac{d\tilde{\mu}}{d\tilde{\nu}}\right) d\tilde{\nu}$$

$$= \alpha \times \int_\Omega \left(\ln \frac{d\mu_1}{d\tilde{\nu}_1}\right) d\tilde{\nu}_1 + (1-\alpha)\int_\Omega \left(\ln \frac{d\mu_2}{d\tilde{\nu}_2}\right) d\tilde{\nu}_2$$

$$= \alpha \ \mathrm{KL}(\mu_1,\tilde{\nu}_1) + (1-\alpha)\ \mathrm{KL}(\mu_2,\tilde{\nu}_2).$$

Application:

$$\mathrm{kl}(p,q) = \mathrm{kl}\left(\frac{1}{N-1}\sum_{j\geq 2} Q_j(A_j),\ \frac{1}{N-1}\sum_{j\geq 2} Q_1(A_j)\right)$$

$$\underset{\substack{\text{joint}\\ \text{convexity}\\ \text{of kl}}}{\leq} \frac{1}{N-1}\sum_{j\geq 2} \mathrm{kl}\big(Q_j(A_j),\ Q_1(A_j)\big)$$

$$\underset{\substack{\text{data-}\\ \text{compression ineq.}}}{\leq} \frac{1}{N-1}\sum_{j\geq 2} \mathrm{KL}(Q_j, Q_1) = \overline{K}.$$

(3) We denote $(k \geq 1)$ $a_k = \mathbb{E}_k\left[\frac{1}{T}\sum_{t=1}^T p_{kt}\right]$ and $b_k = \mathbb{E}_1\left[\frac{1}{T}\sum_{t=1}^T p_{kt}\right]$

note that $b_1 = a_1$

The "2nd point" of the proof in (2) was purely analytical and only used that $b_1 + b_2 + \cdots + b_N = 1$, which is still true.

Therefore, we similarly get:

$$\min_{k=1,\dots N} \mathbb{E}_k\left[\frac{1}{T}\sum_{t=1}^T p_{kt}\right] \leq \max\left\{\frac{2e}{2e+1},\ \mathrm{kl}\left(\frac{1}{N-1}\sum_{j\geq 2}\mathbb{E}_j\left[\frac{1}{T}\sum_t p_{jt}\right],\ \frac{1}{N-1}\sum_{j\geq 2}\mathbb{E}_1\left[\frac{1}{T}\sum_t p_{jt}\right]\right) \times 1/\ln N\right\}$$

By convexity of kl, we may further upper bound the right-hand side by

$$\max\left\{\frac{2e}{2e+1},\ \frac{1}{\ln N}\left(\frac{1}{N-1}\sum_{j=2}^N \mathrm{kl}\left(\mathbb{E}_j\left[\frac{1}{T}\sum_t p_{jt}\right],\ \mathbb{E}_1\left[\frac{1}{T}\sum_t p_{jt}\right]\right)\right)\right\}$$

It thus suffices to show that $\mathrm{kl}(\mathbb{E}_j Z, \mathbb{E}_1 Z) \leq \mathrm{KL}(\mathbb{P}_j^L, \mathbb{P}_1^L)$

for any random variable Z that

- takes values in $[0,1]$
- is $\sigma(L)$-measurable

} that is,

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random variables Z of the form $Z = \Psi(L)$
where $\Psi : \ell = (\ell_{j,t})_{\substack{j \leq N \\ t \leq T}} \longmapsto \Psi(\ell) \in [0,1]$ is measurable.

But
$$\mathbb{E}_j Z = \mathbb{E}_j \Psi(L) = \int \Psi(\ell) \, d\mathbb{P}_j^L(\ell)$$
where $\mathbb{P}_j^L$ is the image distribution of $\mathbb{P}_j$ by L.

The result thus follows from the two reminders of the property of the KL divergence (see first page of the statement of the exercise):

<u>Lemma</u>: Let $f : (\Omega, \mathcal{F}) \longrightarrow [0,1]$ be measurable and let μ, ν be probability distributions over $(\Omega, \mathcal{F})$. Then:
$$\mathrm{kl}\left(\int_\Omega f \, d\mu, \int_\Omega f \, d\nu \right) \leq \mathrm{KL}(\mu, \nu).$$

<u>Proof</u>: Let $\tilde{\Omega} = \Omega \times [0,1]$ (equipped with the product σ-algebra)
Let $E = \{ (\omega, t) \text{ s.t. } f(\omega) \geqslant t \}$; E is measurable

Let $\tilde{\mu} = \mu \otimes d$ and $\tilde{\nu} = \nu \otimes d$ where d is the Lebesgue measure on $[0,1]$

$$\mathrm{KL}(\mu, \nu) = \mathrm{KL}(\tilde{\mu}, \tilde{\nu}) \geqslant \mathrm{kl}\big(\tilde{\mu}(E), \tilde{\nu}(E)\big)$$

cf 1st reminder on KL divergence as $\mathrm{KL}(d,d) = 0$

a special case of data compression with $X = \mathbb{1}_E$

But by Fubini–Tonelli:
$$\tilde{\mu}(E) = \iint \mathbb{1}_{\{f(\omega) \geqslant t\}} \, d\mu(\omega) \, dt = \int f(\omega) \, d\mu(\omega)$$
integrating over t

and similarly for $\tilde{\nu}(E)$.

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(4) By independence, $\mathbb{P}_j^L$ is a product of NT distributions. Using that $KL(\mu \otimes \mu', \nu \otimes \nu') = KL(\mu, \nu) + KL(\mu', \nu')$ (iterating this equality), we get

$$KL(\mathbb{P}_j^L, \mathbb{P}_1^L) = \sum_{k,t} KL(\mathbb{P}_j^{L_{kt}}, \mathbb{P}_1^{L_{kt}})$$

$$\begin{cases} = 0 & \text{if } k \neq 1 \text{ and } k \neq j \\ = KL(\mathrm{Ber}(1/2-\varepsilon), \mathrm{Ber}(1/2)) & \text{if } k = j \\ = KL(\mathrm{Ber}(1/2), \mathrm{Ber}(1/2-\varepsilon)) & \text{if } k = 1 \end{cases}$$

Thus, $\forall j$, $KL(\mathbb{P}_j^L, \mathbb{P}_1^L) = T \times \Big(\underbrace{kl(1/2-\varepsilon, 1/2) + kl(1/2, 1/2-\varepsilon)}_{} \Big)$

it suffices to show that this is $\leq 5\varepsilon^2$ when $\varepsilon \leq 1/10$.

$$\begin{aligned} & kl(1/2-\varepsilon, 1/2) + kl(1/2, 1/2-\varepsilon) \\ &= \left(\frac{1}{2}-\varepsilon\right) \ln \frac{1/2-\varepsilon}{1/2} + \left(1-(1/2-\varepsilon)\right) \ln \frac{1-(1/2-\varepsilon)}{1/2} + \frac{1}{2} \ln \frac{1/2}{1/2-\varepsilon} + \frac{1}{2} \ln \frac{1/2}{1-(1/2-\varepsilon)} \\ &= (1/2-\varepsilon) \ln(1-2\varepsilon) + (1/2+\varepsilon) \ln(1+2\varepsilon) - \frac{1}{2} \ln(1-2\varepsilon) - \frac{1}{2} \ln(1+2\varepsilon) \\ &= \varepsilon \ln(1+2\varepsilon) - \varepsilon \ln(1-2\varepsilon) = \varepsilon \ln\big((1+2\varepsilon)/(1-2\varepsilon) \big) \\ &= \varepsilon \ln\left(1 + \frac{4\varepsilon}{1-2\varepsilon}\right) \underset{\ln(1+u) \leq u}{\leq} \frac{4\varepsilon^2}{1-2\varepsilon} \underset{\substack{\text{for } \varepsilon \leq 1/10, \\ 1-2\varepsilon \geq 4/5}}{\leq} 5\varepsilon^2 \end{aligned}$$

Hence

$$\begin{aligned} \overline{K}' &= \frac{1}{N-1} \sum_{j \geq 2} KL(\mathbb{P}_j^L, \mathbb{P}_1^L) \\ &= T\big(kl(1/2-\varepsilon, 1/2) + kl(1/2, 1/2-\varepsilon) \big) \leq 5T\varepsilon^2 \quad \text{for } \varepsilon \leq 1/10. \end{aligned}$$

(5) Questions (1)-(4) lead to $\forall \varepsilon \in (0, 1/10]$,

$$\begin{aligned} SR_T \overset{\text{def}}{=} \sup_{\ell_{jt} \in [0,1]} \left\{ \sum_{t,j} p_{jt} \ell_{jt} - \min_k \sum_{t=1}^{T} \ell_{kt} \right\} &\geq T\varepsilon \left(1 - \min_k \mathbb{E}_k\left[\frac{1}{T} \sum_t p_{kt} \right] \right) \\ &\geq T\varepsilon \left(1 - \max\left\{ \frac{2e}{2e+1}, \frac{5T\varepsilon^2}{\ln N} \right\} \right) \end{aligned}$$

We would like to take, e.g., ε such that $5T\varepsilon^2/\ln N = \frac{2e}{2e+1}$

that is, $\varepsilon^* = \sqrt{\frac{2e}{2e+1}\frac{\ln N}{5T}}$

This ε^* is $\leq 1/10$ when

$$\frac{\ln N}{T} \leq \frac{(2e+1)5}{2e} \times \frac{1}{100} \approx 0{,}059197$$

and $1/17 \approx 0{,}58823$

Thus, $\varepsilon^* \leq 1/10$ when $T \geq 17 \ln N$.

With this ε^*, the bound becomes $T\,\varepsilon^*\left(1 - \frac{2e}{2e+1}\right)$

$$= \sqrt{T \ln N} \times \underbrace{\left(\sqrt{\frac{2e}{(2e+1)5}} \times \frac{1}{2e+1}\right)}_{\geq 0.06}$$

<u>Theorem</u> For all strategies, for all $N \geq 2$, for all $T \geq 17 \ln N$,

$$\sup_{\ell_{jt} \in [0,1]} \left\{ \sum_{t,j} p_{jt}\,\ell_{jt} - \min_k \sum_t \ell_{kt} \right\} \geq 0.06\sqrt{T \ln N}$$

<u>PS</u> There will be bonus points for those who will significantly improve both constants 17 and 0.06 ! In particular, the 0.06 should become as close as possible to $1/\sqrt{2} \simeq 0{,}7$.