

Solution for Exercise 1:

Pick η_t as $\eta_t = \frac{\gamma}{M-m} \sqrt{\frac{\ln N}{t}}$ where γ is to be determined by the analysis

Hoeffding's lemma: $\delta_t \leq \frac{\eta_t^2}{8} (M-m)^2$

So that:

$$R_T = \sum_{t,j} p_{jt} l_{jt} - \min_k \sum_{t=1}^T l_{kt} \leq \frac{\ln N}{\eta_T} + \sum_{t=1}^T \delta_t \leq \frac{M-m}{\gamma} \sqrt{T \ln N} + \gamma \frac{(M-m)^2}{8} \sum_{t=1}^T \sqrt{\frac{\ln N}{t}}$$

Now, $\sum_{t=1}^T \frac{1}{\sqrt{t}} \leq \int_0^T \frac{1}{\sqrt{t}} dt = [2\sqrt{t}]_0^T = 2\sqrt{T}$

$$R_T \leq (M-m) \sqrt{T \ln N} \left(\frac{1}{\gamma} + \frac{2\gamma}{8} \right) \stackrel{\text{optimal value } \gamma^* = 2}{=} (M-m) \sqrt{T \ln N}$$

We only lose a $\sqrt{2}$ factor w.r.t bound when T is known (with the doubling trick). $\sqrt{2}$ is the price for adaptivity in T . ∇ better constant than doubling

Solution for Exercise 2:

If you find one (even in several weeks!) please send it to me! The reward for a significant contribution will be given by bonus points at the exam.

Solution for Exercise 3.

(1)
$$v_t = \sum_j p_{jt} (l_{jt} - \sum_k p_{kt} l_{kt})^2$$

is some variance
thus, by the variational formula for the variance

$$= \min_{\mu \in \mathbb{R}} \sum_j p_{jt} (l_{jt} - \mu)^2$$

pick $\mu = 0$

$$\leq \sum_j p_{jt} l_{jt}^2$$

thus $l_{jt} \geq 0$ and $l_{jt} \leq M$
thus $l_{jt}^2 \leq M l_{jt}$

$$\leq M \sum_j p_{jt} l_{jt}$$

Hence, summing over $t=1, \dots, T$:

$$\sum_{t=1}^T v_t = \sum_{t=1}^T \sum_{j=1}^N p_{jt} (l_{jt} - \sum_{k=1}^N p_{kt} l_{kt})^2 \leq M \sum_{t=1}^T \sum_{j=1}^N p_{jt} l_{jt}$$

(2) The theorem for EWA tuned with $\eta_t = \frac{\ln N}{\sum_{s=1}^t \delta_s}$ ensures that for this algorithm (since $w=0$):

$$\sum_{t,j} p_{jt} l_{jt} - \min_k \sum_t l_{kt} \leq \underbrace{2\sqrt{\sum_t v_t \ln N}}_{\leq 2\sqrt{M \sum_{t,j} p_{jt} l_{jt} \ln N}} + M \left(2 + \frac{4}{3} \ln N\right) \quad (*)$$

Thus, denoting $x = \sqrt{\sum_{t,j} p_{jt} l_{jt}}$, we have the 2nd order inequality:

$$x^2 \leq (2\sqrt{M \ln N})x + \left(\min_k \sum_t l_{kt} + M \left(2 + \frac{4}{3} \ln N\right) \right)$$

A lemma that we saw in class says that if $x^2 \leq b + a\sqrt{x}$ ($a, b \geq 0$) then $x \leq a + \sqrt{b}$

This means here that

$$\sqrt{\sum_{t,j} p_{jt} l_{jt}} \leq 2\sqrt{M \ln N} + \sqrt{\min_k \sum_t l_{kt} + M \left(2 + \frac{4}{3} \ln N\right)}$$

We could take $()^2$ of both sides

but it is slightly more elegant to substitute this inequality into (*)

$$\leq \sqrt{\min_k \sum_t l_{kt}} + \sqrt{M \left(2 + \frac{4}{3} \ln N\right)}$$

We get:

$$\begin{aligned} \sum_{t,j} p_{jt} \ell_{jt} - \min_k \sum_t \ell_{kt} \\ \leq 4M \ln N + 2M \sqrt{(2 + \frac{4}{3} \ln N) \ln N} + 2 \sqrt{M \min_k \sum_t \ell_{kt} \ln N} \\ + M(2 + \frac{4}{3} \ln N) \end{aligned}$$

(3)

let's make these more readable!

We may assume $\ell_{kt} \geq 0$ $N \geq 2$ in which case $\ln N \geq 0.693$ and $1 \leq \frac{3}{2} \ln N$

$$\begin{aligned} \text{Then, } 4M \ln N + 2M \sqrt{(2 + \frac{4}{3} \ln N) \ln N} + M(2 + \frac{4}{3} \ln N) &\leq M \ln N \times (4 + 2 \sqrt{3 + \frac{4}{3}} + 3 + \frac{4}{3}) \\ &\leq 13M \ln N \end{aligned}$$

$\uparrow \leq 2 \times \frac{3}{2} \ln N$
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$$\text{Final bound: } \sum_{t,j} p_{jt} \ell_{jt} - \min_k \sum_{t=1}^T \ell_{kt} \leq 13M \ln N + 2 \sqrt{M \min_k \sum_t \ell_{kt} \ln N}$$

$\uparrow$
 error suffered
 when
 $\exists k \mid \sum_t \ell_{kt} = 0$
 i.e. all $\ell_{kt} = 0$
 given that losses
 are non-negative

Solution for Exercise 4.

Assume that $\eta \leq 1/2M$: $-\eta \ell_{k,t} \geq -1/2$,
 positive weights $p_{j,t}$, algorithm well-defined

$$\begin{aligned}
 (1) \quad -\eta \sum_{j=1}^N p_{j,t} \ell_{j,t} &\geq \ln(1 - \eta \sum_{j=1}^N p_{j,t} \ell_{j,t}) \\
 &\quad \uparrow \ln(1+u) \leq u \\
 &= \ln\left(\sum_{j=1}^N p_{j,t} (1 - \eta \ell_{j,t})\right) \\
 &= \ln \frac{\sum_{j=1}^N \pi_{j,t-1} (1 - \eta \ell_{j,t})}{\sum_{k \in N} \prod_{s=1}^{t-1} (1 - \eta \ell_{k,s})} \\
 &\quad \uparrow \text{def. of } p_{j,t}
 \end{aligned}$$

Telescoping sum:

$$-\eta \sum_{t=1}^T \sum_{j=1}^N p_{j,t} \ell_{j,t} \geq \ln \frac{\sum_{k \in N} \prod_{t=1}^T (1 - \eta \ell_{k,t})}{N}$$

$\uparrow$
 convention: empty $\prod$ equals 1

We lower bound $\ln\left(\sum_{k \in N} \prod_{t=1}^T (1 - \eta \ell_{k,t})\right)$:

$$\begin{aligned}
 \forall j, \quad &\geq \ln\left(\prod_{t=1}^T (1 - \eta \ell_{j,t})\right) = \sum_{t=1}^T \ln(1 - \eta \ell_{j,t}) \\
 &\geq -\eta \ell_{j,t} - \eta^2 \ell_{j,t}^2 \\
 &\quad \begin{array}{l} -\eta \ell_{j,t} \geq -1/2 \rightarrow \\ \text{and} \\ \ln(1+u) \geq u - u^2 \quad \forall u \geq -1/2 \end{array}
 \end{aligned}$$

Summarizing what we got so far:

$$\forall j, \quad -\eta \sum_{t=1}^T \sum_{k=1}^N p_{k,t} \ell_{k,t} \geq -\ln N - \eta \sum_{t=1}^T \ell_{j,t} - \eta^2 \sum_{t=1}^T \ell_{j,t}^2$$

$$\text{Thus, } \forall j, \quad \sum_{t=1}^T \sum_{k=1}^N p_{k,t} \ell_{k,t} - \sum_{t=1}^T \ell_{j,t} \leq \frac{\ln N}{\eta} + \eta \sum_{t=1}^T \ell_{j,t}^2.$$

We did so under the assumption that $\forall j,t$, $-\eta \ell_{j,t} \geq -1/2$
 valid as soon as $\ell_{j,t} \leq M$ and $\eta \leq 1/2M$.

(2) Waiting for your solutions! (But keep in mind that it's a difficult problem...)