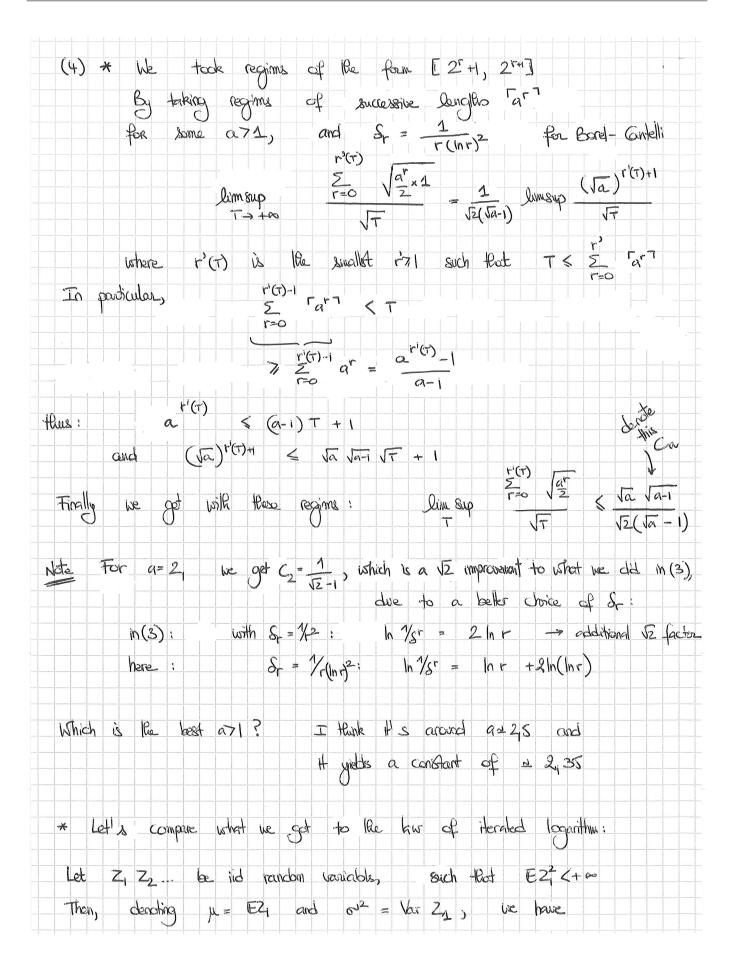


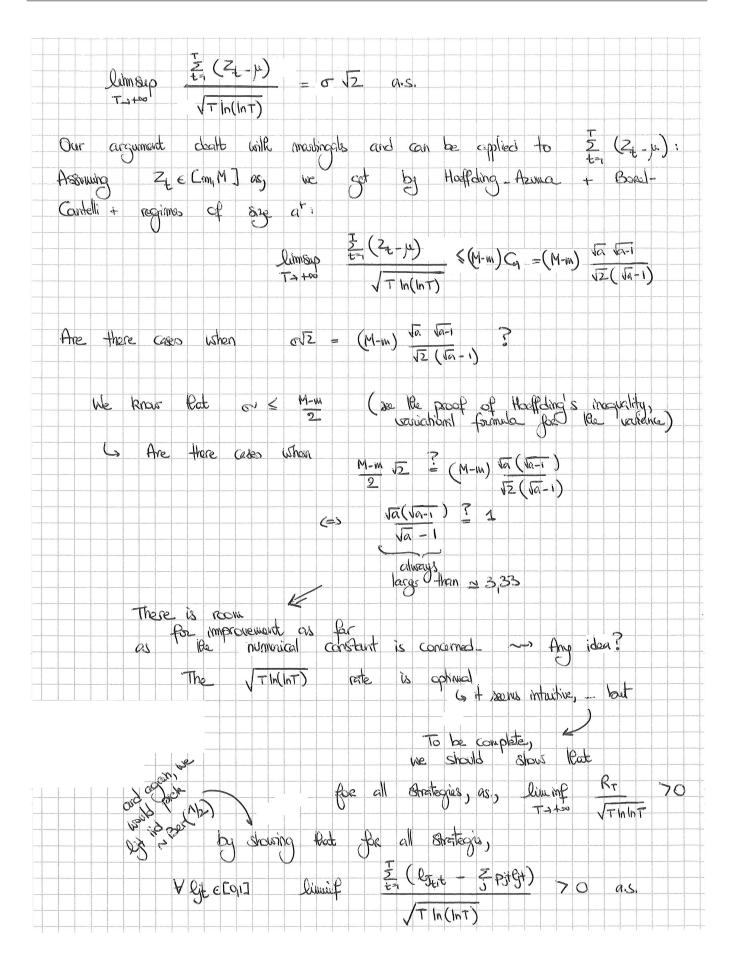
For question (4) I provide two answers:

- My original answer, where I perform a doubling trick with regimes of lengths given by the integer part of a<sup>r</sup> instead of 2<sup>r</sup>; the constant may be improved but I explain why we still have a gap w.r.t. law of the iterated logarithm
- An answer by Dau Hai Dang (a student who took the course in Spring 2019), where he explains how a modification of the Borel-Cantelli lemma, based on a doubling trick (!), does the job

This all should be some food for thought!

And maybe a clearer summary can be written (also with lower bounds). Please send me your notes if they are worth it!





SPDG, supposons que M-m = 1 et on souhaite controler demastrer que

$$\lim_{t \to +\infty} \frac{S_t}{\sqrt{t \log \log t}} \leq C =: \frac{1}{\sqrt{2}} \quad p.s. \quad (1)$$
  
$$S_t = \sum_{\lambda=1}^{t} \left( l_{J_{\lambda,\lambda}} - \mathbb{E}[l_{J_{\lambda,\lambda}} | F_{\lambda-1}] \right).$$

Rappelons que par l'inégalité de Pool, on a

oū

$$\mathbb{P}\left(\sup_{\substack{t\leq T}} S_t \ge \varepsilon\right) \le \exp\left(-\frac{2\varepsilon^2}{T}\right). \tag{2}$$

Maintenant, fixens un 2 >0 et posens Vt l'événement suivant

$$V_t = \left\{ \sup_{t \in S_t} S_t \leq (C + \varepsilon) \sqrt{t \log \log t} \right\}.$$

Lemme (Borel-Castelli modifié) Pour démontrer (1), il suffit de démontrer que, pour tout 270 et pour un a>1 quelonque, on a

$$\sum_{x} \mathbb{P}\left[\left(V_{\lfloor a^{x}+2 \rfloor} \cap V_{\lfloor a^{x}+2 \rfloor} \cap \cdots \cap V_{\lfloor a^{x}+2 \rfloor}\right)^{c}\right] < +\infty$$

Où B<sup>c</sup> signifie le complément de l'évenement B.

Preure du lemme (exactement comme la preure de Borel- Contelli).

-'inegalité' signifie que 
$$\blacksquare \mathbb{E}\left[\sum_{n} \mathbb{I}\left(V_{\lfloor n^{n+1}\rfloor} \cap \cdots \cap V_{\lfloor n^{n+2}\rfloor}\right)^{c}\right] < +\infty$$
  
donc  $\sum_{n} \mathbb{I}\left(V_{\lfloor n^{n+2}\rfloor} \cap \cdots \cap V_{\lfloor n^{n+1}\rfloor}\right)^{c} < +\infty$  ps  
 $\Rightarrow \forall n > n(\omega): \omega \in V_{\lfloor n^{n}+1\rfloor} \cap \cdots \cap V_{\lfloor n^{n+2}\rfloor}$   
 $\Rightarrow \forall t > t_{O}(\omega): \omega \in V_{t}. On conclut var de hinthan de Yt  $\Box$$ 

Retour à la preuve de (1)

$$\begin{split} & \mathbb{P}\Big[\left(\mathbb{V}_{[a^{n}+1]} \cap \mathbb{V}_{[a^{n}+2]} \cap \cdots \cap \mathbb{V}_{[a^{n+1}]}\right)^{c}\Big] \\ &= \mathbb{P}\Big(\left[\exists t: [a^{r}+4] \leq t \leq [a^{r+4}] tq S_{t} > (c+\epsilon)\sqrt{tligligt}\right) \\ &\leq \mathbb{P}\Big(\left[\exists t: [a^{r}+1] \leq t \leq [a^{r+1}] tq S_{t} > (c+\epsilon)\sqrt{a^{n}liglig(a^{n})}\right) \\ &\leq \mathbb{P}\Big(\left[\exists t: [a^{r}+1] \leq t \leq [a^{r+1}] tq S_{t} > (c+\epsilon)\sqrt{a^{n}liglig(a^{n})}\right) \\ &\leq \mathbb{P}\Big(\left[\exists t: [a^{r}+1] \leq t \leq [a^{r+1}] tq S_{t} > (c+\epsilon)\sqrt{a^{n}liglig(a^{n})}\right) \\ &\stackrel{(2)}{\leq} \exp\left(-\frac{2(c+\epsilon)^{2}a^{n}liglig(a^{n})}{[a^{n+1}]}\right) \leq \exp\left(-\frac{2(c+\epsilon)^{2}a^{n}liglig(a^{n})}{a^{n+1}}\right) \end{split}$$

$$\leq \exp\left(-\frac{2\left((+\epsilon)^{2}\log(n\log a)\right)}{a}\right) = \exp\left(-\frac{2\left((+\epsilon)^{2}\log\log a\right)}{a}\right) - \frac{2\left((+\epsilon)^{2}\log\log a\right)}{a}\right) + \frac{2\left((+\epsilon)^{2}\log\log a\right)}{a}$$

Il suffit danc de chuinir a >1 tel que

 $\sum_{\lambda} \lambda^{-2} \frac{(C+E)^2}{a} < +\infty.$ 

Or, comme  $C = \frac{1}{\sqrt{2}}$ , un tel a existe toujours

Remarque. La constante  $C = \frac{1}{\sqrt{2}}$  est optimale, comme vous avez dit dans le consigé.