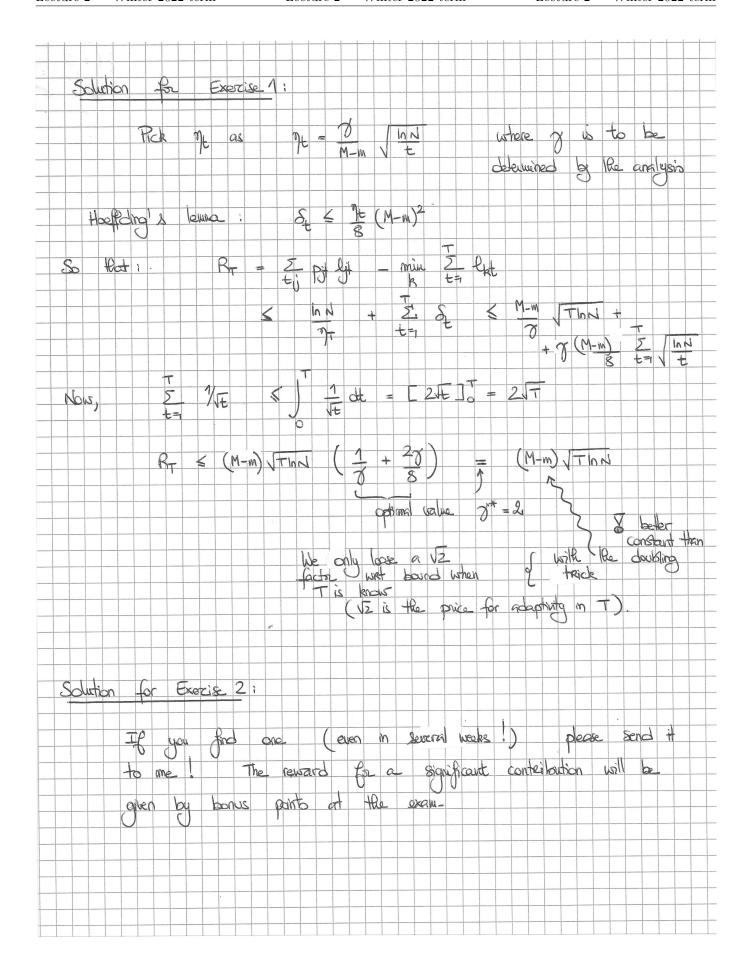
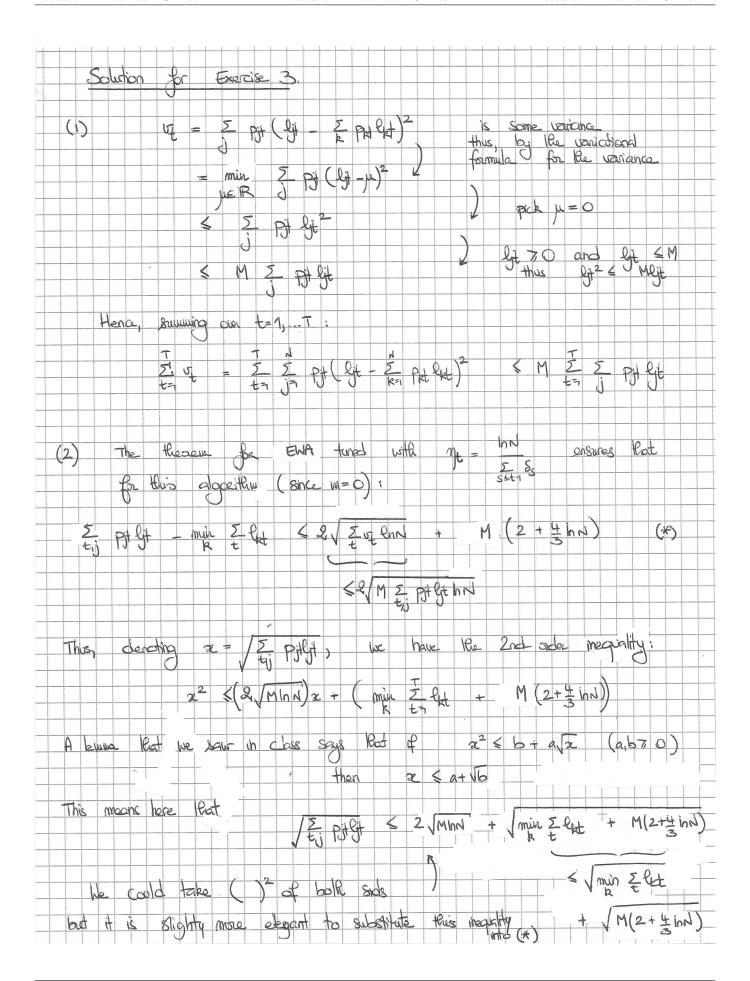
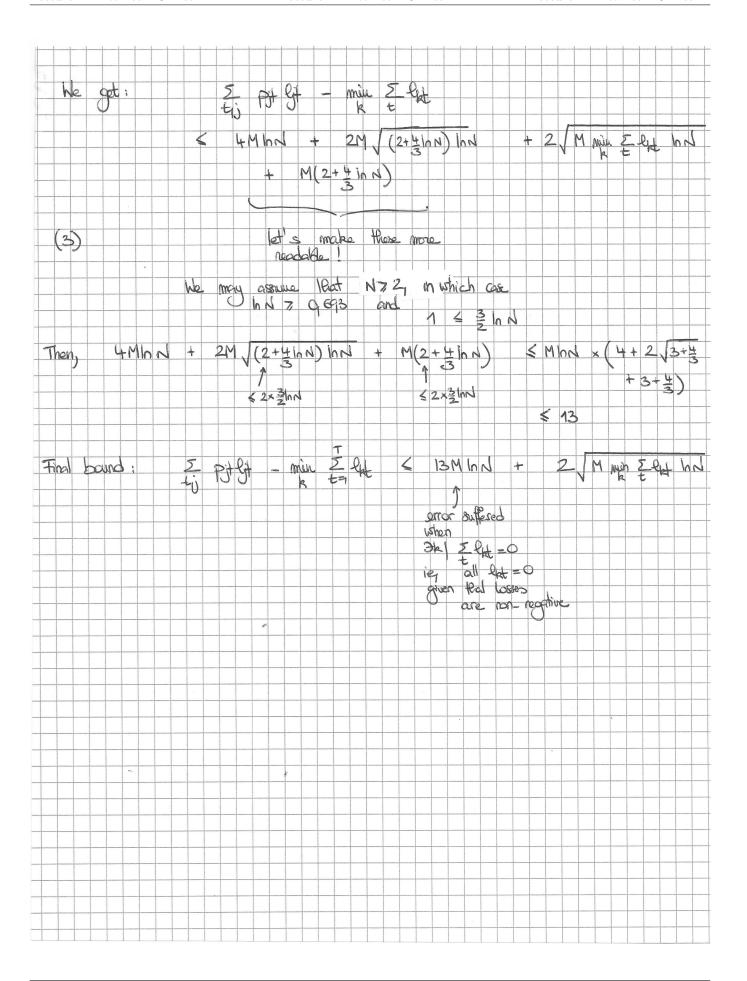
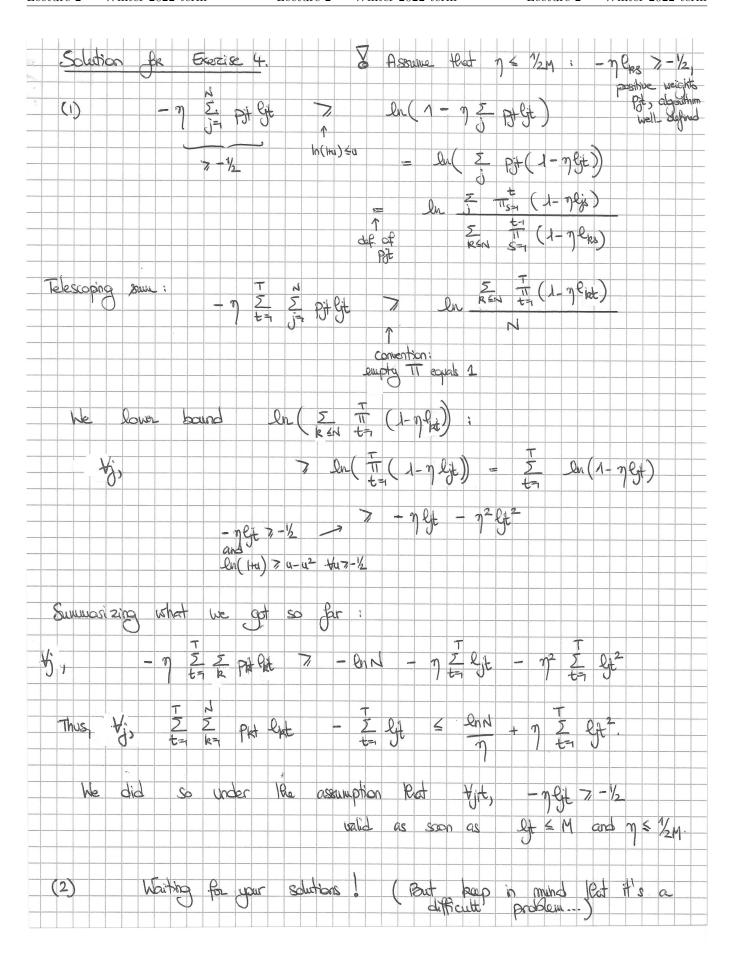
Correction of the four exercises around calibration of EWA

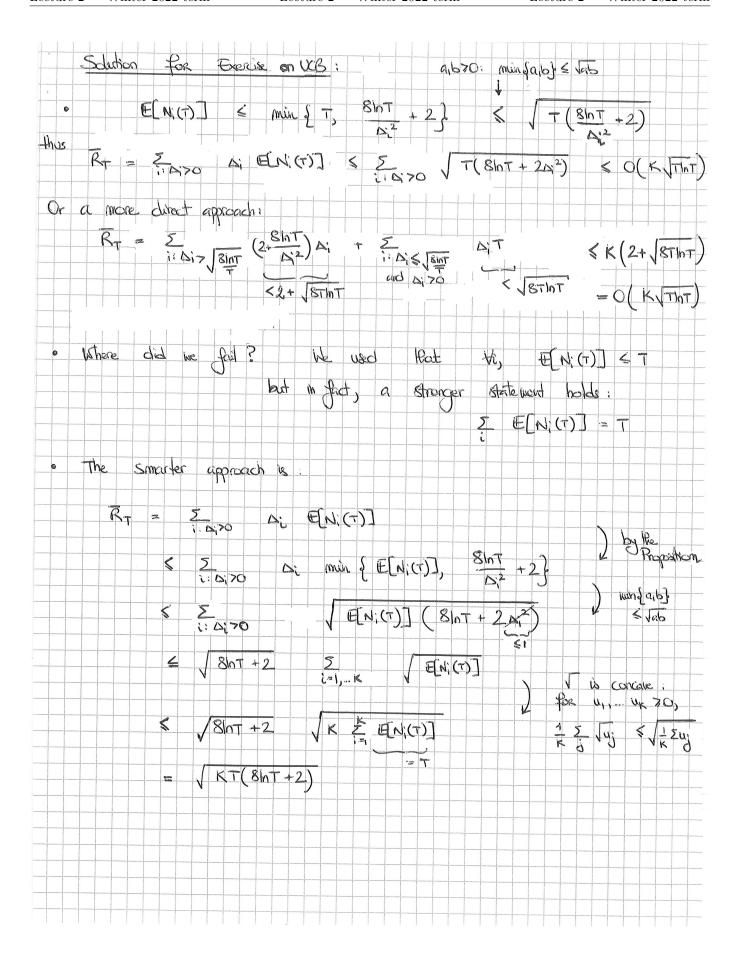


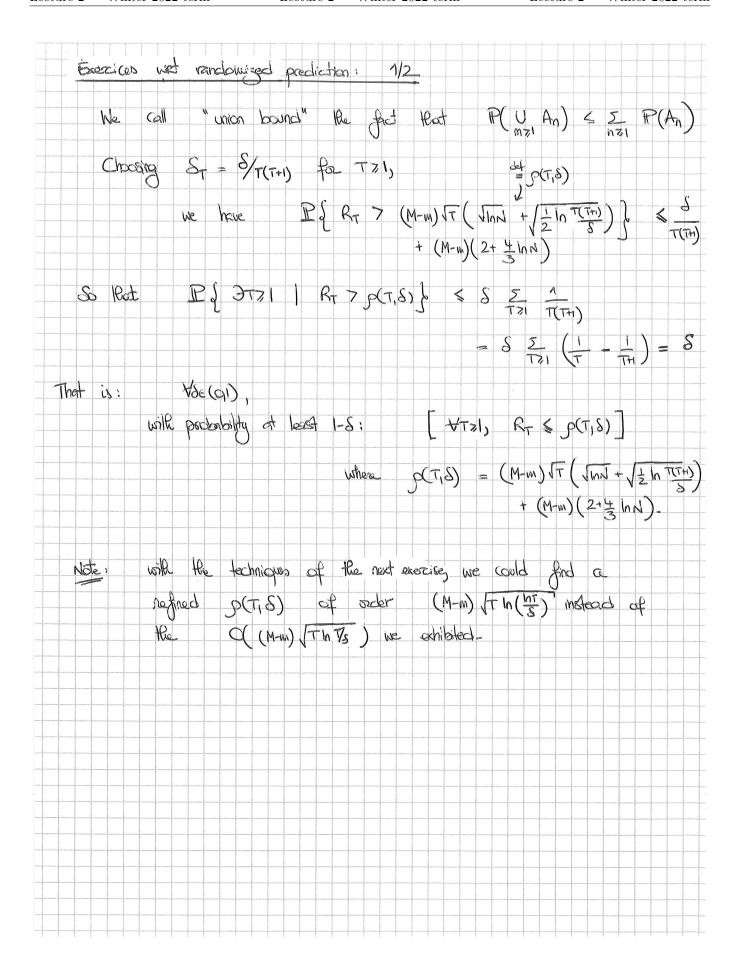




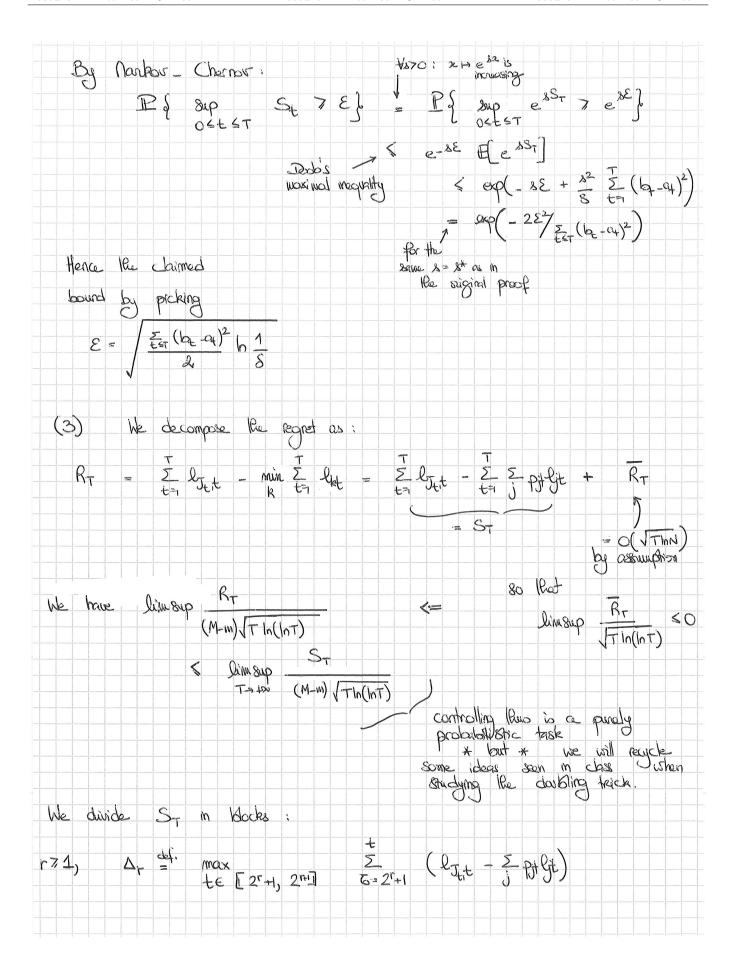


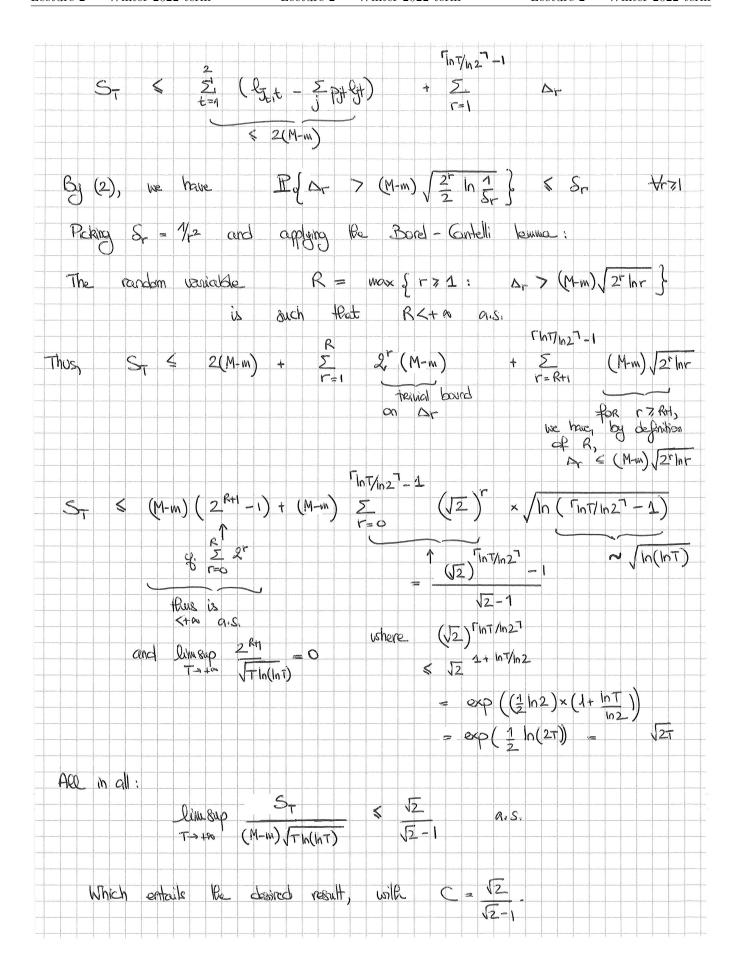
Correction of exercises around randomized prediction





| Exect les    | wet randomize                       | d prediction:         | 2/2               |                        |  |
|--------------|-------------------------------------|-----------------------|-------------------|------------------------|--|
| (1)          |                                     | guen a 8<br>(St.)(70) |                   |                        | n (24)   |
| - (          | St) + 70 is a                       | mantingle who         | eh ¥0<6           | . <τ, X <sub>t</sub> = | EX, IF.  |
|              | St) + 70 is a                       | Submantingale         | when              | X,L s                  | SE[XT   FE]                                    |
| 1000         | St) + 70 is (                       | a Supermorbingal      | <u>ishen</u>      | X <sub>E</sub> 3       | $\mathbb{E}(x_1   \mathcal{F}_{\mathbf{t}})$   |
|              | ditional Jensen!.<br>Submoutingale. |                       |                   |                        |  |
| <u>6x</u> :  | if (St)tro                          | is a massin           | gle thour subm    | mantrigile, fe         | n all seR.                                     |
| Dab's a      | naximal inequality                  | for non-no            | phile Submouti    | ngals (St)tza          |  |
|              | 4770, YC>                           | 0,                    | Pf sup<br>Off < T | St 7 Cf                | < E[S,]  |
| - 08 - ton A | Rodera culomb                       |                       |                   |                        | 70 exists:                                     |
|              | ¥c70,                               | P{ 840 + 70           | St 7 C}           | ( ES.]                 |  |
| (2) W        | the the notation                    | 1 1                   | U                 | G55 ;                  | t  |
| ù            | s a martingale                      | co where              | S <sub>t</sub> =  | 2 X = 2                | t  |
| 29 as        | t tse R,                            | (e sst) +70           | s is a ma         | on-nagotive su         | omostingle.                                    |
| We s         | enoused in class (                  | by induction)         | Pot E             | ess, ] < exp(          | $\frac{3^2}{8}$ $\sum_{t=1}^{7} (b_t - a_t)^2$ |



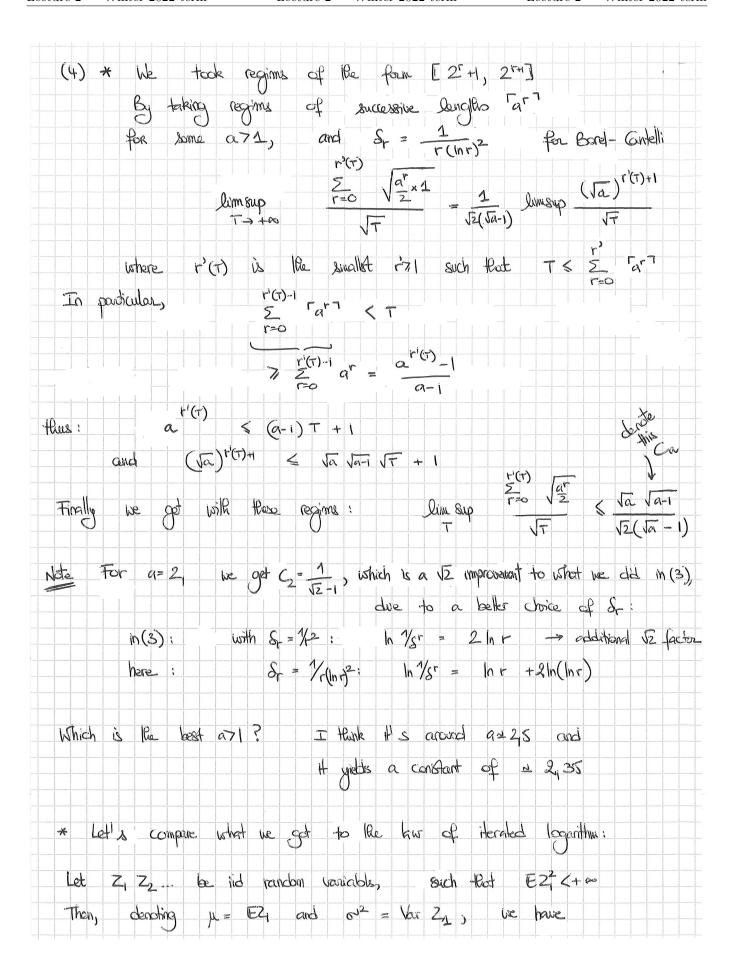


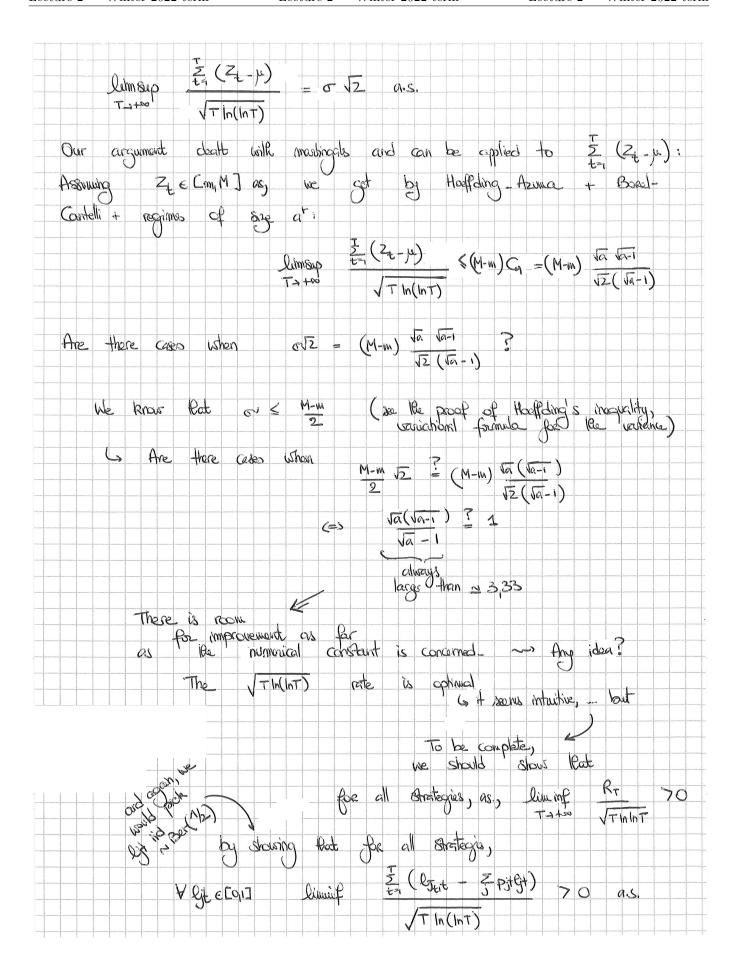
For question (4) I provide two answers:

- My original answer, where I perform a doubling trick with regimes of lengths given by the integer part of a<sup>r</sup> instead of 2<sup>r</sup>; the constant may be improved but I explain why we still have a gap w.r.t. law of the iterated logarithm
- An answer by Dau Hai Dang (a student who took the course in Spring 2019), where he explains how a modification of the Borel-Cantelli lemma, based on a doubling trick (!), does the job

This all should be some food for thought!

And maybe a clearer summary can be written (also with lower bounds). Please send me your notes if they are worth it!





Constante optimale pour la borne du regret

\2

SPDG, supposons que M-m = 1 et on souhaite controler demantrer que

$$\lim_{t\to +\infty} \frac{St}{\sqrt{t \log \log t}} \leq C = : \frac{1}{\sqrt{2}} \text{ p.s.} \quad (1)$$
où  $St = \sum_{s=2}^{t} \left( \ell_{J_{s,s}} - \mathbb{E}[\ell_{J_{s,s}} | \mathcal{F}_{s-4}] \right).$ 

Rappelons que par l'inégolité de Port, on a

$$\mathbb{P}\left(\sup_{t\leq T} S_{t} \geq \varepsilon\right) \leq \exp\left(-\frac{2\varepsilon^{2}}{T}\right). \tag{2}$$

Maintenant, fixons un E > 0 et posons Vt l'évenement suivant

$$Vt = \left\{ S_t \leq (C+E) \sqrt{t \log \log t} \right\}.$$

Lemme (Borel-Cartelli modifié) Pour démontrer (1), il suffit de démontrer que,

pour tout £70 et pour un a>1 quelconque, on a

Où BC signifie le complément de l'évenement B.

Preure du lemme (exactement comme la preuve de Borel-Contelli).

Retour à la preuve de (1)

$$\begin{aligned} & \mathbb{P}\left[\left(Y_{|a^{1}+1}\right) \cap V_{|a^{1}+2}\right) \cap \dots \cap V_{|a^{1}+1}\right]^{c}\right] \\ & = \mathbb{P}\left[\left(\exists t: |a^{r}+1| \le t \le |a^{r+1}| \right) tq \mid S_{t} > (c+\epsilon)\sqrt{t \log \log \left(a^{n}\right)}\right) \\ & \leq \mathbb{P}\left(\left(\exists t: |a^{r}+1| \le t \le |a^{r+1}| \right) tq \mid S_{t} > (c+\epsilon)\sqrt{a^{r} \log \log \left(a^{n}\right)}\right) \\ & \leq \mathbb{P}\left(\left(\exists t: |a^{r}+1| \le t \le |a^{r+1}| \right) tq \mid S_{t} > (c+\epsilon)\sqrt{a^{r} \log \log \left(a^{n}\right)}\right) \\ & \stackrel{(2)}{\leq} \exp\left(-\frac{2(c+\epsilon)^{2} a^{r} \log \log \left(a^{r}\right)}{|a^{r}+1|}\right) \leq \exp\left(-\frac{2(c+\epsilon)^{2} a^{r} \log \log \left(a^{n}\right)}{a^{n+1}}\right) \end{aligned}$$

 $\Box$ 

$$\leq \exp\left(-\frac{2((t+\epsilon)^2\log(n\log a)}{a}\right) = \exp\left(-\frac{2((t+\epsilon)^2\log\log a)}{a}\right) \frac{-2((t+\epsilon)^2}{a}$$

Il suffit danc de choisir a > 1 tel que

$$\sum_{\lambda} x^{-2(C+\epsilon)^2} < +\infty.$$

Or, comme 
$$C = \frac{1}{\sqrt{2}}$$
, un tel a existe toujours

Remarque. La constante  $C = \frac{1}{\sqrt{2}}$  est optimale, comme vous avez dit dans le corrigé.