Correction of the four exercises around calibration of EWA

Solution for Exercise 1 :

Pick η_t as $\eta_t = \dfrac{\gamma}{M-m}\sqrt{\dfrac{\ln N}{t}}$ where γ is to be determined by the analysis

Hoeffding's lemma : $\delta_t \leq \dfrac{\eta_t}{8}(M-m)^2$

So that :
$$R_T = \sum_{t,j} p_{jt}\, \ell_{jt} - \min_k \sum_{t=1}^T \ell_{kt}$$
$$\leq \frac{\ln N}{\eta_T} + \sum_{t=1}^T \delta_t \leq \frac{M-m}{\gamma}\sqrt{T \ln N} + \gamma \frac{(M-m)}{8} \sum_{t=1}^T \sqrt{\frac{\ln N}{t}}$$

Now,
$$\sum_{t=1}^T 1/\sqrt{t} \leq \int_0^T \frac{1}{\sqrt{t}}\,dt = \left[2\sqrt{t}\right]_0^T = 2\sqrt{T}$$

$$R_T \leq (M-m)\sqrt{T\ln N}\ \underbrace{\left(\frac{1}{\gamma} + \frac{2\gamma}{8}\right)}_{\text{optimal value } \gamma^* = 2} = (M-m)\sqrt{T \ln N}$$

better constant than with the doubling trick

We only lose a $\sqrt{2}$ factor wrt bound when T is know
($\sqrt{2}$ is the price for adaptivity in T).

Solution for Exercise 2 :

If you find one (even in several weeks!) please send it to me! The reward for a significant contribution will be given by bonus points at the exam.

 Lecture 2 — Winter 2022 term Lecture 2 — Winter 2022 term

Solution for Exercise 3.

(1)
$$v_t = \sum_j p_{jt}\Big(\ell_{jt} - \sum_k p_{kt}\ell_{kt}\Big)^2$$
is some variance
thus, by the variational formula for the variance
$$= \min_{\mu\in\mathbb{R}} \sum_j p_{jt}(\ell_{jt}-\mu)^2$$
pick $\mu = 0$
$$\le \sum_j p_{jt}\,\ell_{jt}^2$$
$\ell_{jt} \geq 0$ and $\ell_{jt} \le M$ thus $\ell_{jt}^2 \le M\ell_{jt}$
$$\le M \sum_j p_{jt}\,\ell_{jt}$$

Hence, summing over $t=1,\dots T$:
$$\sum_{t=1}^{T} v_t = \sum_{t=1}^{T}\sum_{j=1}^{N} p_{jt}\Big(\ell_{jt} - \sum_{k=1}^{N} p_{kt}\ell_{kt}\Big)^2 \le M \sum_{t=1}^{T}\sum_j p_{jt}\,\ell_{jt}$$

(2) The theorem for EWA tuned with $\eta_t = \dfrac{\ln N}{\sum_{s\le t-1}\delta_s}$ ensures that for this algorithm (since $m=0$) :
$$\sum_{t,j} p_{jt}\ell_{jt} - \min_k \sum_t \ell_{kt} \le 2\sqrt{\underbrace{\sum_t v_t \ln N}} + M\Big(2+\frac{4}{3}\ln N\Big) \qquad (*)$$
$$\le 2\sqrt{M\sum_{t,j} p_{jt}\ell_{jt}\ln N}$$

Thus, denoting $x = \sqrt{\sum_{t,j} p_{jt}\ell_{jt}}$, we have the 2nd order inequality:
$$x^2 \le \big(2\sqrt{M\ln N}\big)x + \Big(\min_k \sum_{t=1}^{T}\ell_{kt} + M\Big(2+\frac{4}{3}\ln N\Big)\Big)$$

A lemma that we saw in class says that if $x^2 \le b + a\sqrt{x}$ $(a,b\geq 0)$
then $x \le a + \sqrt{b}$

This means here that
$$\sqrt{\sum_{t,j} p_{jt}\ell_{jt}} \le 2\sqrt{M\ln N} + \sqrt{\underbrace{\min_k \sum_t \ell_{kt} + M\Big(2+\frac{4}{3}\ln N\Big)}}$$
$$\le \sqrt{\min_k \sum_t \ell_{kt}} + \sqrt{M\Big(2+\frac{4}{3}\ln N\Big)}$$

We could take $(\ )^2$ of both sides
but it is slightly more elegant to substitute this inequality into $(*)$

We get: $\sum_{t,j} p_{jt} \ell_{jt} - \min_k \sum_t \ell_{kt}$

$$\leq 4M\ln N + 2M\sqrt{\left(2+\frac{4}{3}\ln N\right)\ln N} + 2\sqrt{M \min_k \sum_t \ell_{kt} \ln N}$$
$$\underbrace{+ M\left(2+\frac{4}{3}\ln N\right)}_{\text{let's make these more readable!}}$$

(3)

We may assume that $N \geq 2$, in which case $\ln N \geq 0{,}693$ and $1 \leq \frac{3}{2}\ln N$

Then, $4M\ln N + 2M\sqrt{\left(\underbrace{2}_{\leq 2\times\frac{3}{2}\ln N}+\frac{4}{3}\ln N\right)\ln N} + M\left(\underbrace{2}_{\leq 2\times\frac{3}{2}\ln N}+\frac{4}{3}\ln N\right) \leq M\ln N \times \left(4 + 2\sqrt{3+\frac{4}{3}} + 3+\frac{4}{3}\right)$

≤ 13

Final bound: $\sum_{t,j} p_{jt}\ell_{jt} - \min_k \sum_{t=1}^{T} \ell_{kt} \leq 13M\ln N + 2\sqrt{M \min_k \sum_t \ell_{kt} \ln N}$

error suffered when $\exists k \mid \sum_t \ell_{kt} = 0$ ie, all $\ell_{kt} = 0$ given that losses are non-negative

 Lecture 2 — Winter 2022 term Lecture 2 — Winter 2022 term

Solution for Exercise 4.

Assume that $\eta \leq 1/2M$: $-\eta \ell_{ks} \geq -1/2$, positive weights p_{jt}, algorithm well defined

(1)
$$\underbrace{-\eta \sum_{j=1}^{N} p_{jt}\ell_{jt}}_{\geq -1/2} \underset{\ln(1+u)\leq u}{\geq} \ln\Big(1-\eta\sum_{j} p_{jt}\ell_{jt}\Big) = \ln\Big(\sum_j p_{jt}(1-\eta\ell_{jt})\Big)$$
$$\underset{\text{def. of } p_{jt}}{=} \ln \frac{\sum_j \prod_{s=1}^{t}(1-\eta\ell_{js})}{\sum_{k\leq N}\prod_{s=1}^{t-1}(1-\eta\ell_{ks})}$$

Telescoping sum:
$$-\eta\sum_{t=1}^{T}\sum_{j=1}^{N} p_{jt}\ell_{jt} \underset{\substack{\text{convention:}\\ \text{empty } \prod \text{ equals } 1}}{\geq} \ln\frac{\sum_{k\leq N}\prod_{t=1}^{T}(1-\eta\ell_{kt})}{N}$$

We lower bound $\ln\Big(\sum_{k\leq N}\prod_{t=1}^{T}(1-\eta\ell_{kt})\Big)$:

$\forall j$,
$$\geq \ln\Big(\prod_{t=1}^{T}(1-\eta\ell_{jt})\Big) = \sum_{t=1}^{T}\ln(1-\eta\ell_{jt})$$
$$\geq -\eta\ell_{jt} - \eta^2\ell_{jt}^2$$

$-\eta\ell_{jt} \geq -1/2$ and $\ln(1+u) \geq u-u^2 \ \forall u \geq -1/2$

Summarizing what we got so far:

$\forall j$,
$$-\eta\sum_{t=1}^{T}\sum_{k} p_{kt}\ell_{kt} \geq -\ln N - \eta\sum_{t=1}^{T}\ell_{jt} - \eta^2\sum_{t=1}^{T}\ell_{jt}^2$$

Thus, $\forall j$,
$$\sum_{t=1}^{T}\sum_{k=1}^{N} p_{kt}\ell_{kt} - \sum_{t=1}^{T}\ell_{jt} \leq \frac{\ln N}{\eta} + \eta\sum_{t=1}^{T}\ell_{jt}^2.$$

We did so under the assumption that $\forall j,t$, $-\eta\ell_{jt} \geq -1/2$
valid as soon as $\ell_{jt} \leq M$ and $\eta \leq 1/2M$.

(2) Waiting for your solutions! (But keep in mind that it's a difficult problem...)

Correction of exercises around randomized prediction

Solution for Exercise on UCB:

$a,b>0$: $\min\{a,b\} \leq \sqrt{ab}$

- $$\mathbb{E}[N_i(T)] \leq \min\left\{T, \frac{8\ln T}{\Delta_i^2}+2\right\} \leqslant \sqrt{T\left(\frac{8\ln T}{\Delta_i^2}+2\right)}$$

thus
$$\overline{R}_T = \sum_{i:\Delta_i>0} \Delta_i \,\mathbb{E}[N_i(T)] \leqslant \sum_{i:\Delta_i>0} \sqrt{T(8\ln T + 2\Delta_i^2)} \leqslant O\left(K\sqrt{T\ln T}\right)$$

Or a more direct approach:
$$\overline{R}_T = \sum_{i:\Delta_i>\sqrt{\frac{8\ln T}{T}}} \underbrace{\left(2+\frac{8\ln T}{\Delta_i^2}\right)\Delta_i}_{<2+\sqrt{8T\ln T}} + \sum_{\substack{i:\Delta_i\leqslant\sqrt{\frac{8\ln T}{T}} \\ \text{and } \Delta_i>0}} \underbrace{\Delta_i T}_{<\sqrt{8T\ln T}} \leqslant K\left(2+\sqrt{8T\ln T}\right) = O\left(K\sqrt{T\ln T}\right)$$

- Where did we fail? We used that $\forall i$, $\mathbb{E}[N_i(T)] \leq T$ but in fact, a stronger statement holds:
$$\sum_i \mathbb{E}[N_i(T)] = T$$

- The smarter approach is:

$$\begin{aligned}
\overline{R}_T &= \sum_{i:\Delta_i>0} \Delta_i \,\mathbb{E}[N_i(T)] \\
&\leqslant \sum_{i:\Delta_i>0} \Delta_i \min\left\{\mathbb{E}[N_i(T)], \frac{8\ln T}{\Delta_i^2}+2\right\} && \text{by the Proposition} \\
&\leqslant \sum_{i:\Delta_i>0} \sqrt{\mathbb{E}[N_i(T)]\,\big(8\ln T + 2\underbrace{\Delta_i^2}_{\leq 1}\big)} && \min\{a,b\} \leqslant \sqrt{ab} \\
&\leq \sqrt{8\ln T+2} \sum_{i=1,\ldots K} \sqrt{\mathbb{E}[N_i(T)]} && \sqrt{\ } \text{ is concave: for } u_1,\ldots u_K\geqslant 0,\ \frac{1}{K}\sum_j \sqrt{u_j} \leqslant \sqrt{\frac{1}{K}\sum u_j} \\
&\leqslant \sqrt{8\ln T+2}\,\sqrt{K\underbrace{\sum_{i=1}^K \mathbb{E}[N_i(T)]}_{=T}} \\
&= \sqrt{KT(8\ln T+2)}
\end{aligned}$$

Exercices wrt randomized prediction : 1/2

We call "union bound" the fact that $\mathbb{P}\left(\bigcup_{n \geq 1} A_n\right) \leq \sum_{n \geq 1} \mathbb{P}(A_n)$

Choosing $\delta_T = \delta / T(T+1)$ for $T \geq 1$,

we have
$$\mathbb{P}\Big\{ R_T > \underbrace{(M-m)\sqrt{T}\left(\sqrt{\ln N} + \sqrt{\frac{1}{2}\ln\frac{T(T+1)}{\delta}}\right) + (M-m)\left(2 + \frac{4}{3}\ln N\right)}_{\stackrel{\text{def}}{=} \rho(T,\delta)} \Big\} \leq \frac{\delta}{T(T+1)}$$

So that
$$\mathbb{P}\left\{ \exists T \geq 1 \mid R_T > \rho(T,\delta) \right\} \leq \delta \sum_{T \geq 1} \frac{1}{T(T+1)} = \delta \sum_{T \geq 1} \left(\frac{1}{T} - \frac{1}{T+1} \right) = \delta$$

That is: $\forall \delta \in (0,1)$,

with probability at least $1-\delta$: $\left[\forall T \geq 1, \ R_T \leq \rho(T,\delta) \right]$

where
$$\rho(T,\delta) = (M-m)\sqrt{T}\left(\sqrt{\ln N} + \sqrt{\frac{1}{2}\ln\frac{T(T+1)}{\delta}}\right) + (M-m)\left(2 + \frac{4}{3}\ln N\right).$$

Note: with the techniques of the next exercise, we could find a refined $\rho(T,\delta)$ of order $(M-m)\sqrt{T \ln\left(\frac{\ln T}{\delta}\right)}$ instead of the $\mathcal{O}\left((M-m)\sqrt{T \ln T/\delta}\right)$ we exhibited.

Exercises wrt randomized prediction: 2/2

(1) Recall that given a filtration $(\mathcal{F}_t)_{t\geq 0}$, and given an adapted process $(S_t)_{t\geq 0}$, we say that:

- $(S_t)_{t\geq 0}$ is a martingale when $\forall\, 0\leq t\leq T$, $X_t = \mathbb{E}[X_T \mid \mathcal{F}_t]$
- $(S_t)_{t\geq 0}$ is a submartingale when $X_t \leq \mathbb{E}[X_T \mid \mathcal{F}_t]$
- $(S_t)_{t\geq 0}$ is a supermartingale when $X_t \geq \mathbb{E}[X_T \mid \mathcal{F}_t]$

By the conditional Jensen's inequality, a convex function of a martingale is a submartingale.

Ex: if $(S_t)_{t\geq 0}$ is a martingale then $(|S_t|)_{t\geq 0}$ and $(\exp(s\,S_t))_{t\geq 0}$ are submartingales, for all $s\in\mathbb{R}$.

Doob's maximal inequality for non-negative submartingale $(S_t)_{t\geq 0}$:

$$\forall T\geq 0,\quad \forall c>0,\qquad \mathbb{P}\Big\{\sup_{0\leq t\leq T} S_t \geq c\Big\} \leq \frac{\mathbb{E}[S_T]}{c}$$

A not-so-famous version for non-negative supermartingale $(S_t)_{t\geq 0}$ exists:

$$\forall c>0,\qquad \mathbb{P}\Big\{\sup_{t\geq 0} S_t \geq c\Big\} \leq \frac{\mathbb{E}[S_0]}{c}$$

(2) With the notation of the proof given in class:

$(S_t)_{t\geq 0}$ where $S_t = \sum_{\tau=1}^{t} X_\tau - \sum_{\tau=1}^{t} \mathbb{E}[X_\tau \mid \mathcal{F}_{\tau-1}]$ is a martingale $(S_0 = 0)$

so that $\forall s\in\mathbb{R}$, $(e^{sS_t})_{t\geq 0}$ is a non-negative submartingale.

We proved in class (by induction) that $\mathbb{E}[e^{sS_T}] \leq \exp\Big(\frac{s^2}{8}\sum_{t=1}^{T}(b_t-a_t)^2\Big)$

By Markov – Chernov:

$$\mathbb{P}\Big\{ \sup_{0\leq t\leq T} S_t \geq \varepsilon \Big\} \overset{\substack{\forall s>0:\ x\mapsto e^{sx} \text{ is increasing}}}{=} \mathbb{P}\Big\{ \sup_{0\leq t\leq T} e^{sS_t} \geq e^{s\varepsilon} \Big\}$$

$$\underset{\text{Doob's maximal inequality}}{\leq} e^{-s\varepsilon}\, \mathbb{E}\big[e^{sS_T}\big]$$

$$\leq \exp\Big(-s\varepsilon + \frac{s^2}{8} \sum_{t=1}^{T} (b_t - a_t)^2 \Big)$$

$$\underset{\text{for the same } s = s^* \text{ as in the original proof}}{=} \exp\Big(-2\varepsilon^2 \Big/ \sum_{t\leq T} (b_t - a_t)^2 \Big)$$

Hence the claimed bound by picking

$$\varepsilon = \sqrt{\frac{\sum_{t\leq T} (b_t - a_t)^2}{2} \ln\frac{1}{\delta}}$$

(3) We decompose the regret as:

$$R_T = \sum_{t=1}^{T} \ell_{J_t,t} - \min_k \sum_{t=1}^{T} \ell_{k,t} = \underbrace{\sum_{t=1}^{T} \ell_{J_t,t} - \sum_{t=1}^{T} \sum_j p_{j,t}\ell_{j,t}}_{= S_T} + \underbrace{\overline{R}_T}_{\substack{= O(\sqrt{T\ln N}) \\ \text{by assumption}}}$$

We have $\displaystyle\limsup \frac{R_T}{(M-m)\sqrt{T\ln(\ln T)}}$ $\Leftarrow$ so that $\displaystyle\limsup \frac{\overline{R}_T}{\sqrt{T\ln(\ln T)}} \leq 0$

$$\leq \limsup_{T\to+\infty} \frac{S_T}{(M-m)\sqrt{T\ln(\ln T)}}$$

controlling this is a purely probabilistic task
* but * we will recycle some ideas seen in class when studying the doubling trick.

We divide S_T in blocks:

$$r \geq 1, \qquad \Delta_r \overset{\text{def.}}{=} \max_{t\in[2^r+1,\, 2^{r+1}]} \sum_{\tau=2^r+1}^{t} \Big(\ell_{J_\tau,\tau} - \sum_j p_{j,\tau}\ell_{j,\tau} \Big)$$

$$S_T \leqslant \underbrace{\sum_{t=1}^{2} \Big(\ell_{J_t,t} - \sum_j p_{j,t}\ell_{j,t} \Big)}_{\leqslant\, 2(M-m)} + \sum_{r=1}^{\lceil \ln T/\ln 2 \rceil - 1} \Delta_r$$

By (2), we have $\mathbb{P}\left\{ \Delta_r > (M-m)\sqrt{\frac{2^r}{2}\ln\frac{1}{\delta_r}} \right\} \leqslant \delta_r \qquad \forall r \geqslant 1$

Picking $\delta_r = 1/r^2$ and applying the Borel-Cantelli lemma:

The random variable $R = \max\left\{ r \geqslant 1 : \ \Delta_r > (M-m)\sqrt{2^r \ln r} \right\}$

is such that $R < +\infty$ a.s.

Thus, $$S_T \leqslant 2(M-m) + \sum_{r=1}^{R} \underbrace{2^r (M-m)}_{\substack{\text{trivial bound}\\ \text{on } \Delta_r}} + \sum_{r=R+1}^{\lceil \ln T/\ln 2 \rceil - 1} \underbrace{(M-m)\sqrt{2^r \ln r}}_{\substack{\text{for } r \geqslant R+1,\\ \text{we have, by definition}\\ \text{of } R,\\ \Delta_r \leqslant (M-m)\sqrt{2^r \ln r}}}$$

$$S_T \leqslant \underbrace{(M-m)\big(2^{R+1}-1\big)}_{\substack{\text{this is}\\ <+\infty \text{ a.s.}}} + (M-m) \underbrace{\sum_{r=0}^{\lceil \ln T/\ln 2 \rceil - 1} \big(\sqrt{2}\big)^r}_{= \frac{(\sqrt{2})^{\lceil \ln T/\ln 2 \rceil} - 1}{\sqrt{2}-1}} \times \sqrt{\ln \underbrace{\big(\lceil \ln T/\ln 2 \rceil - 1 \big)}_{\sim \sqrt{\ln(\ln T)}}}$$

$2^{R+1}-1$: cf. $\displaystyle\sum_{r=0}^{R} 2^r$

and $\displaystyle\limsup_{T\to+\infty} \frac{2^{R+1}}{\sqrt{T \ln(\ln T)}} = 0$

where $$\big(\sqrt{2}\big)^{\lceil \ln T/\ln 2 \rceil} \leqslant \sqrt{2}^{\,1+\ln T/\ln 2} = \exp\left(\left(\tfrac{1}{2}\ln 2\right) \times \left(1 + \frac{\ln T}{\ln 2}\right) \right) = \exp\left(\tfrac{1}{2}\ln(2T)\right) = \sqrt{2T}$$

All in all:

$$\limsup_{T\to+\infty} \frac{S_T}{(M-m)\sqrt{T\ln(\ln T)}} \leqslant \frac{\sqrt{2}}{\sqrt{2}-1} \quad \text{a.s.}$$

Which entails the desired result, with $C = \dfrac{\sqrt{2}}{\sqrt{2}-1}$.

For question (4) I provide two answers:

- My original answer, where I perform a doubling trick with regimes of lengths given by the integer part of a^r instead of 2^r; the constant may be improved but I explain why we still have a gap w.r.t. law of the iterated logarithm
- An answer by Dau Hai Dang (a student who took the course in Spring 2019), where he explains how a modification of the Borel-Cantelli lemma, based on a doubling trick (!), does the job

This all should be some food for thought!

And maybe a clearer summary can be written (also with lower bounds). Please send me your notes if they are worth it!

(4) * We took regimes of the form $[2^r+1, 2^{r+1}]$

By taking regimes of successive lengths $\lceil a^r \rceil$ for some $a > 1$, and $\delta_r = \dfrac{1}{r(\ln r)^2}$ for Borel-Cantelli

$$\limsup_{T \to +\infty} \frac{\sum_{r=0}^{r'(T)} \sqrt{\frac{a^r}{2} \times 1}}{\sqrt{T}} = \frac{1}{\sqrt{2}(\sqrt{a}-1)} \limsup \frac{(\sqrt{a})^{r'(T)+1}}{\sqrt{T}}$$

where $r'(T)$ is the smallest $r' \geq 1$ such that $T \leq \sum_{r=0}^{r'} \lceil a^r \rceil$

In particular,
$$\underbrace{\sum_{r=0}^{r'(T)-1} \lceil a^r \rceil}_{\geq \sum_{r=0}^{r'(T)-1} a^r = \frac{a^{r'(T)}-1}{a-1}} < T$$

thus: $a^{r'(T)} \leq (a-1)T + 1$

and $(\sqrt{a})^{r'(T)+1} \leq \sqrt{a}\sqrt{a-1}\sqrt{T} + 1$

Finally we get with these regimes: $$\limsup_T \frac{\sum_{r=0}^{r'(T)} \sqrt{\frac{a^r}{2}}}{\sqrt{T}} \leq \frac{\sqrt{a}\sqrt{a-1}}{\sqrt{2}(\sqrt{a}-1)}$$ denote this C_a

<u>Note</u> For $a=2$, we get $C_2 = \dfrac{1}{\sqrt{2}-1}$, which is a $\sqrt{2}$ improvement to what we did in (3), due to a better choice of δ_r:

in (3): with $\delta_r = 1/r^2$: $\ln 1/\delta^r = 2 \ln r \rightarrow$ additional $\sqrt{2}$ factor

here: $\delta_r = 1/r(\ln r)^2$: $\ln 1/\delta^r = \ln r + 2\ln(\ln r)$

Which is the best $a > 1$? I think it's around $a \simeq 2{,}5$ and it yields a constant of $\simeq 2{,}35$

* Let's compare what we got to the law of iterated logarithm:

Let $Z_1, Z_2, \ldots$ be iid random variables, such that $\mathbb{E} Z_1^2 < +\infty$

Then, denoting $\mu = \mathbb{E} Z_1$ and $\sigma^2 = \operatorname{Var} Z_1$, we have

$$\limsup_{T\to+\infty} \frac{\sum_{t=1}^{T} (Z_t-\mu)}{\sqrt{T\ln(\ln T)}} = \sigma\sqrt{2} \quad \text{a.s.}$$

Our argument dealt with martingales and can be applied to $\sum_{t=1}^{T} (Z_t-\mu)$: Assuming $Z_t \in [m, M]$ a.s., we get by Hoeffding–Azuma + Borel–Cantelli + regimes of size a^r:

$$\limsup_{T\to+\infty} \frac{\sum_{t=1}^{T} (Z_t-\mu)}{\sqrt{T\ln(\ln T)}} \leqslant (M-m)\,C_a = (M-m)\,\frac{\sqrt{a}\,\sqrt{a-1}}{\sqrt{2}(\sqrt{a}-1)}$$

Are there cases when $\sigma\sqrt{2} = (M-m)\,\dfrac{\sqrt{a}\,\sqrt{a-1}}{\sqrt{2}\,(\sqrt{a}-1)}$?

We know that $\sigma \leq \dfrac{M-m}{2}$ (see the proof of Hoeffding's inequality, variational formula for the variance)

↳ Are there cases when

$$\frac{M-m}{2}\sqrt{2} \overset{?}{=} (M-m)\,\frac{\sqrt{a}\,(\sqrt{a-1})}{\sqrt{2}(\sqrt{a}-1)}$$

$$\iff \underbrace{\frac{\sqrt{a}(\sqrt{a-1})}{\sqrt{a}-1}}_{\text{always larger than } \simeq 3{,}33} \overset{?}{=} 1$$

There is room for improvement as far as the numerical constant is concerned. ⇝ Any idea?

The $\sqrt{T\ln(\ln T)}$ rate is optimal

↳ it seems intuitive, … but

To be complete, we should show that

for all strategies, a.s., $\displaystyle\liminf_{T\to+\infty} \frac{R_T}{\sqrt{T\ln\ln T}} > 0$

and again, we would pick $\ell_{j,t}$ iid $\sim$ Ber(1/2)

by showing that for all strategies,

$$\forall\, \ell_{j,t} \in [0,1] \qquad \liminf \frac{\sum_{t=1}^{T} \left(\ell_{J_t,t} - \sum_j p_{j,t}\ell_{j,t}\right)}{\sqrt{T\ln(\ln T)}} > 0 \quad \text{a.s.}$$

 Lecture 2 — Winter 2022 term Lecture 2 — Winter 2022 term

Constante optimale pour la borne du regret

\2

SPDG, supposons que $M - m = 1$ et on souhaite ~~contrôler~~ démontrer que

$$\limsup_{t \to +\infty} \frac{S_t}{\sqrt{t \log \log t}} \le C =: \frac{1}{\sqrt{2}} \quad \text{p.s.} \qquad (1)$$

où $S_t = \sum_{s=1}^{t} \left(\ell_{J_s,s} - \mathbb{E}[\ell_{J_s,s} \mid \mathcal{F}_{s-1}] \right)$.

Rappelons que par l'inégalité de Doob, on a

$$\mathbb{P}\left(\sup_{t \le T} S_t \ge \varepsilon \right) \le \exp\left(- \frac{2\varepsilon^2}{T} \right). \qquad (2).$$

Maintenant, fixons un $\varepsilon > 0$ et posons V_t l'évènement suivant

$$V_t = \left\{ \cancel{\sup} \; S_t \le (C+\varepsilon)\sqrt{t \log\log t} \right\}.$$

<u>Lemme</u> (Borel-Cantelli modifié) Pour démontrer (1), il suffit de démontrer que, pour tout $\varepsilon > 0$ et pour un $a > 1$ quelconque, on a

$$\sum_r \mathbb{P}\left[\left(V_{\lfloor a^r + 1 \rfloor} \cap V_{\lfloor a^r + 2 \rfloor} \cap \dots \cap V_{\lfloor a^{r+1} \rfloor} \right)^c \right] < +\infty$$

où B^c signifie le complément de l'évènement B.

<u>Preuve du lemme</u> (exactement comme la preuve de Borel-Cantelli).

L'inégalité signifie que $\mathbb{E}\left[\sum_r \mathbb{1}_{(V_{\lfloor a^r+1 \rfloor} \cap \dots \cap V_{\lfloor a^{r+1} \rfloor})^c} \right] < +\infty$

donc $\sum_r \mathbb{1}_{(V_{\lfloor a^r+1 \rfloor} \cap \dots \cap V_{\lfloor a^{r+1} \rfloor})^c} < +\infty$ ps

$\Rightarrow \forall r > r_0(\omega) : \omega \in V_{\lfloor a^r+1 \rfloor} \cap \dots \cap V_{\lfloor a^{r+1} \rfloor}$

$\Rightarrow \forall t > t_0(\omega) : \omega \in V_t$. On conclut par définition de V_t □

<u>Retour à la preuve de (1).</u>

$$\mathbb{P}\left[\left(V_{\lfloor a^r+1 \rfloor} \cap V_{\lfloor a^r+2 \rfloor} \cap \dots \cap V_{\lfloor a^{r+1} \rfloor} \right)^c \right]$$

$$= \mathbb{P}\left(\exists t : \lfloor a^r + 1 \rfloor \le t \le \lfloor a^{r+1} \rfloor \text{ tq } S_t > (C+\varepsilon)\sqrt{t \log\log t} \right)$$

$$\le \mathbb{P}\left(\exists t : \lfloor a^r + 1 \rfloor \le t \le \lfloor a^{r+1} \rfloor \text{ tq } S_t > (C+\varepsilon)\sqrt{a^r \log\log(a^r)} \right)$$

$$\le \mathbb{P}\left(\exists t : \cancel{\lfloor a^r + 1 \rfloor} \; 1 \le t \le \lfloor a^{r+1} \rfloor \text{ tq } S_t > (C+\varepsilon)\sqrt{a^r \log\log(a^r)} \right)$$

$$\overset{(2)}{\le} \exp\left(- \frac{2(C+\varepsilon)^2 a^r \log\log(a^r)}{\lfloor a^{r+1} \rfloor} \right) \le \exp\left(- \frac{2(C+\varepsilon)^2 a^r \log\log(a^r)}{a^{r+1}} \right)$$

 Lecture 2 — Winter 2022 term Lecture 2 — Winter 2022 term

$$\leq \exp\left(-\frac{2(C+\varepsilon)^2\log(r\log a)}{a}\right) = \exp\left(-\frac{2(C+\varepsilon)^2\log\log a}{a}\right) r^{-\frac{2(C+\varepsilon)^2}{a}} \quad 3$$

Il suffit donc de choisir $a > 1$ tel que

$$\sum_r r^{-\frac{2(C+\varepsilon)^2}{a}} < +\infty .$$

Or, comme $C = \frac{1}{\sqrt{2}}$, un tel a existe toujours $\square$.

<u>Remarque</u>. La constante $C = \frac{1}{\sqrt{2}}$ est optimale, comme vous avez dit dans le corrigé.