Exercise 4: Approachability of a closed convex set $\mathcal{C}$

A statistician plays against an opponent; the statistician wants her average loss to approach (converge to) a given closed convex set $\mathcal{C} \subseteq \mathbb{R}^d$, while the opponent aims to prevent this convergence. Formally, the statistician and the opponent have respective action sets $\{1, \ldots, N\}$ and $\{1, \ldots, M\}$ and a loss function

$$\ell : \{1, \ldots, N\} \times \{1, \ldots, M\} \longrightarrow \mathbb{R}^d$$

is given and known by both players. The learning protocol is the following.

Protocol: For all rounds $t = 1, 2, \ldots$,

- the statistician and the opponent simultaneously and independently pick actions $I_t \in \{1, \ldots, N\}$ and $J_t \in \{1, \ldots, M\}$, possibly at random, according to distributions denoted by $\boldsymbol{p}_t$ and $\boldsymbol{q}_t$, respectively;
- the statistician suffers the loss $\ell(I_t, J_t)$;
- both players observe I_t and J_t.

Respective aims: The statistician wants to ensure that

$$\frac{1}{T} \sum_{t=1}^{T} \ell(I_t, J_t) \longrightarrow \mathcal{C} \quad \text{a.s.,} \qquad \text{that is,} \qquad \min_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^{T} \ell(I_t, J_t) \right\| \longrightarrow 0 \quad \text{a.s.,} \tag{1}$$

while the opponent wants to prevent this convergence, i.e., ensure that

$$\mathbb{P} \left\{ \limsup_{T \to \infty} \min_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^{T} \ell(I_t, J_t) \right\| > 0 \right\} > 0 \tag{2}$$

A set $\mathcal{C}$ such that the statistician has a strategy ensuring (1) is called approachable by the statistician. Otherwise, in the case (2), we say that it is not approachable.

Blackwell's condition: We denote by $\mathcal{P}_N$ and $\mathcal{P}_M$ the sets of probability distributions over $\{1, \ldots, N\}$ and $\{1, \ldots, M\}$, respectively. We (bi-)linearly extend ℓ by defining, for all $\boldsymbol{p} = (p_1, \ldots, p_N) \in \mathcal{P}_N$, all $j \in \{1, \ldots, M\}$, and all $\boldsymbol{q} = (q_1, \ldots, q_M) \in \mathcal{P}_M$,

$$\ell(\boldsymbol{p}, j) = \sum_{i=1}^{N} p_i \, \ell(i, j) \qquad \text{and} \qquad \ell(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{N} \sum_{j=1}^{M} p_i \, q_j \, \ell(i, j)$$

We consider Blackwell's condition:

$$\forall \boldsymbol{q} \in \mathcal{P}_M, \ \ \exists \boldsymbol{p} \in \mathcal{P}_N \ \big| \ \ \ell(\boldsymbol{p}, \boldsymbol{q}) \in \mathcal{C} \,,$$

and will show that it is a necessary and sufficient condition for approachability.

Necessity (requires only Lecture #2)

1. Show that when Blackwell's condition does not hold, then not only is $\mathcal{C}$ not approachable by the statistician, but we even have that there exists $\gamma > 0$ such that for all strategies of the statistician,

$$\liminf_{T \to \infty} \min_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^{T} \ell(I_t, J_t) \right\| \geqslant \gamma \quad \text{a.s.}$$

2. Rephrase the previous result in terms of approachability of some set for the opponent.

Hints: For Question 1, show that there exists $\boldsymbol{q}_0 \in \mathcal{P}_M$ such that

$$\min_{\boldsymbol{p} \in \mathcal{P}_N} \min_{c \in \mathcal{C}} \big\| c - \ell(\boldsymbol{p}, \boldsymbol{q}) \big\| > 0$$

and carefully also explain why, for all strategies of the statistician and of the opponent,

$$\left\| \frac{1}{T} \sum_{t=1}^{T} \ell(I_t, J_t) - \frac{1}{T} \sum_{t=1}^{T} \ell(\boldsymbol{p}_t, \boldsymbol{q}_t) \right\| \longrightarrow 0 \quad \text{a.s.}$$

Sufficiency (requires Lecture #3)

We henceforth assume that Blackwell's condition holds and consider the following strategy for the statistician, where we denote by $\langle \cdot\,, \cdot \rangle$ the inner product in $\mathbb{R}^d$.

Strategy for the statistician:
- Play $\boldsymbol{p}_1 = (1/N, \ldots, 1/N)$
- For $t \geqslant 2$,
  - Compute the current average loss $\overline{m}_{t-1} = \dfrac{1}{t-1} \displaystyle\sum_{s=1}^{t-1} \ell(\boldsymbol{p}_s, J_s)$
  - Project it onto $\mathcal{C}$ as $\overline{c}_{t-1} = \Pi_{\mathcal{C}}\big(\overline{m}_{t-1}\big)$
  - Pick $p_t \in \displaystyle\mathop{\arg\min}_{\boldsymbol{p} \in \mathcal{P}_N} \max_{\boldsymbol{q} \in \mathcal{P}_M} \big\langle \overline{m}_{t-1} - \overline{c}_{t-1},\, \ell(\boldsymbol{p}, \boldsymbol{q}) \big\rangle$
  - Draw I_t at random according to $\boldsymbol{p}_t$

We then analyze this strategy; we denote $L = \max_{i,j} \big|\ell(i,j)\big|$.

3. Recall thanks to a picture (no formal proof required) why for all $t \geqslant 2$,

$$\forall c \in \mathcal{C}, \qquad \big\langle \overline{m}_{t-1} - \overline{c}_{t-1},\, c - \overline{c}_{t-1} \big\rangle \leqslant 0$$

4. Deduce from this and from Sion's lemma (the fact that under some conditions, an inf sup equals a sup inf) that

$$\forall \boldsymbol{q} \in \mathcal{P}_M, \qquad \big\langle \overline{m}_{t-1} - \overline{c}_{t-1},\, \ell(\boldsymbol{p}_t, \boldsymbol{q}) - \overline{c}_{t-1} \big\rangle \leqslant 0$$

5. Show that the distance to $\mathcal{C}$ at round t, namely, $d_t = \displaystyle\inf_{c \in \mathcal{C}} \big\| \overline{m}_t - c \big\|$, satisfies, for all $t \geqslant 1$,

$$d_{t+1}^2 \leqslant \left(1 - \frac{1}{t+1}\right)^2 d_t^2 + \frac{4L^2}{(t+1)^2}$$

Hint: consider $c = \overline{c}_t$ and upper bound d_{t+1} by $\big\| \overline{m}_{t+1} - \overline{c}_t \big\|$. Then "decompose" $\overline{m}_{t+1}$ into $\overline{m}_t$ and $\ell(\boldsymbol{p}_{t+1}, J_{t+1})$.

6. Prove that for all $T \geqslant 1$,

$$\min_{c \in \mathcal{C}} \left\| c - \frac{1}{T} \sum_{t=1}^{T} \ell(\boldsymbol{p}_t, J_t) \right\| \leqslant \frac{2L}{\sqrt{T}}\,.$$

7. Conclude. (Yes, there is a simple but final step to deal with.)