Comment tirer parti de l’embarras du choix face à plusieurs modèles de prévision concurrents

Gilles Stoltz

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The framework of this talk

Sequential and worst-case deterministic prediction of time series based on expert advice
A statistician has to predict a time series $y_1, y_2, \ldots \in C$, where $C$ is a convex subset of $\mathbb{R}^d$.

Finitely many expert forecasts are available, e.g., given by some stochastic models.

At each instance $t$, expert $j \in \{1, \ldots, N\}$ outputs a forecast

$$f_{j,t} = f_{j,t}(y_{1}^{t-1}) \in C$$

Observations and predictions are made in a sequential fashion:

The prediction $\hat{y}_t$ of $y_t$ is determined based

- on the past observations $y_{1}^{t-1} = (y_1, \ldots, y_{t-1})$,
- and the current and past expert forecasts $f_{j,s}$, where $s \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, N\}$,

before getting to know the actual value $y_t$. 
A typical solution of the problem is to form convex (or linear) combinations of the expert forecasts, with weights $p_t = (p_{1,t}, \ldots, p_{N,t})$ or $v_t = (v_{1,t}, \ldots, v_{N,t})$ adjusted over time.

The statistician then outputs the forecasts $\hat{y}_t = \sum_{j=1}^{N} p_{j,t} f_{j,t}$

The observations $y_t$ will not be considered stochastic anymore at this stage; thus the performance criterion will be a relative one.

We consider a convex loss function $\ell : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_+$, e.g., the square loss $\ell(x, y) = (x - y)^2$ when $\mathcal{C} \subseteq \mathbb{R}$.

The cumulative losses of the statistician and of the constant convex combinations $q = (q_1, \ldots, q_N)$ of the expert forecasts equal

$$\hat{L}_T = \sum_{t=1}^{T} \ell \left( \sum_{j=1}^{N} p_{j,t} f_{j,t}, y_t \right) \quad \text{and} \quad L_T(q) = \sum_{t=1}^{T} \ell \left( \sum_{j=1}^{N} q_{j} f_{j,t}, y_t \right)$$
First study: Forecasting of air quality

Starting date: September 2005

Academic partner: Vivien Mallet, INRIA, project-team CLIME

Industrial partner: Edouard Debry, INERIS (Institut National de l’EnviRonnement Industriel et des RisqueS)

M.Sc. students involved over time:
- Boris Mauricette (6 months in 2007; from M2 Pro Paris-Diderot and ENS de Lyon)
- Sébastien Gerchinovitz (5 months in 2008; from M2 Maths Paris-Sud)
- Karim Drifi (4 months in 2009; from M2 MVA ENS Cachan)
- Paul Baudin (4 months in 2012; from M2 MVA ENS Cachan)

Associated publication: in the Journal of Geophysical Research
Some characteristics of one among the studied data sets:

- 126 days during summer ’01; one-day ahead prediction

- 241 stations in France and Germany

- Typical ozone concentrations between 40 $\mu$g m$^{-3}$ and 150 $\mu$g m$^{-3}$; sometimes above the values 180 $\mu$g m$^{-3}$ or 240 $\mu$g m$^{-3}$

- 48 experts, built in Mallet et Sportisse ’06 by choosing a physical and chemical formulation, a numerical approximation scheme to solve the involved PDEs, and a set of input data (among many)

→ Instead of trusting only one model/expert (“selection”), we proceed in a more greedy way and consider many models/experts, which we combine sequentially (“aggregation”).

This leads to more accurate and more stable (meta-)predictions.
The stations of the network are indexed by $S$.

Each model $j = 1, \ldots, 48$ outputs a prediction $f_{j,t}^s$ for the ozone peak at station $s$ and day $t$, which is then compared to the measured peak $y_t^s$. (We discard measurement errors.)

The statistician chooses at each round a single convex weight vector $p_t$ or linear weight vector $v_t$ to be used at all stations; this leads to prediction fields.

The strategies are assessed based on their RMSEs, which amounts to considering the convex losses

$$
\ell_t(p_t) \overset{\text{def}}{=} \sum_{s \in S_t} \left( \sum_{j=1}^{48} p_{j,t} f_{j,t}^s - y_t^s \right)^2
$$

where $S_t$ is the subset of active stations at day $t$.

The RMSE equals

$$
\sqrt{\frac{\sum_{t=t_0}^{T} \ell_t(p_t)}{\sum_{t=t_0}^{T} |S_t|}} \text{ for } t_0 = 31 \text{ (short training)}
$$
**Left:** There are several good and useful experts.

**Right:** Their forecasting profiles are quite different (the experts are not clones of the others!).

**Left:** Coloring of Europe according to the index of the locally best expert

**Right:** Average forecasting profiles during a day (averages over time and space)
The framework of this talk
(continued)
The regret $R_T$ is defined as the difference

$$\hat{L}_T - \min_q L_T(q) = \sum_{t=1}^{T} \ell \left( \sum_{j=1}^{N} p_j, f_j, t, y_t \right) - \min_q \sum_{t=1}^{T} \ell \left( \sum_{j=1}^{N} q_j, f_j, t, y_t \right)$$

We are interested in aggregation rules with (uniformly) vanishing per-round regret,

$$\limsup_{T \to \infty} \frac{1}{T} \sup \left\{ \hat{L}_T - \min_q L_T(q) \right\} \leq 0$$

where the supremum is over all possible sequences of observations and of expert forecasts. (Not just over most of these sequences!)

Remarks:

- Hence the name “prediction of individual sequences” (or robust aggregation of expert forecasts).

- The best convex combination $q^*$ is known in hindsight whereas the statistician has to predict in a sequential fashion.
This framework leads to a *meta-statistical* interpretation:

- each series of *expert* forecasts is given by a *statistical* forecasting method, possibly tuned with some given set of parameters;
- these base forecasts relying on some stochastic model are then *combined* in a *robust* and *deterministic* manner.

The *cumulative loss* of the statistician can be decomposed as

\[
\hat{L}_T = \min_{q} L_T(q) + R_T
\]

This leads to the following interpretations:

- the term indicating the performance of the best convex combination of the expert forecasts is an *approximation error*;
- the regret term measures a *sequential estimation error*. 
First study, continued

Forecasting of the air quality
How good are our experts? See the “oracles” below.

Do we expect the aggregation methods to provide significant improvements? Yes, whenever the best convex and/or linear combinations significantly outperform the best expert.

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Performance, in terms of RMSE, of (some combinations of) the experts
Disclaimer

We could also consider batch learning methods to aggregate models/experts, like
- BMA (Bayesian model averaging),
- CART (classification and regression trees),
- random forests, etc.,

or even selection methods, and apply them online, by running a batch analysis at each step.

We instead resort to “real” online techniques that, in addition, come up with theoretical guarantees even in non-stochastic scenarios.

We will also see that calibrating their parameters can be done in a more satisfactory way, using the sequential character of the prediction.
A strategy to pick convex weights

Let’s do some maths!
Reminder of the aim and setting:

Given a loss function $\ell : C \times C \to \mathbb{R}$, where $C \subseteq \mathbb{R}^d$ is convex

Choose sequentially the convex weights $p_{j,t}$

To uniformly bound the regret with respect to all sequences of observations $y_t$ and expert predictions $f_{j,t}$:

$$\sum_{t=1}^{T} \ell \left( \sum_{j=1}^{N} p_{j,t} f_{j,t}, y_t \right) - \min_{q} \sum_{t=1}^{T} \ell \left( \sum_{j=1}^{N} q_{j} f_{j,t}, y_t \right)$$

When $\ell$ is convex in its first argument, sub-gradients exist, i.e.:

For all $x, y \in C$, there exists $\nabla \ell(x, y)$ such that

$$\forall x' \in C, \quad \ell(x, y) - \ell(x', y) \leq \nabla \ell(x, y) \cdot (x - x')$$
To uniformly bound the regret with respect to all convex weight vectors $q$, we write

$$\max_q \sum_{t=1}^T \ell \left( \sum_{j=1}^N p_{j,t} f_{j,t}, y_t \right) - \sum_{t=1}^T \ell \left( \sum_{j=1}^N q_j f_{j,t}, y_t \right) \leq \max_q \sum_{t=1}^T \nabla \ell \left( \sum_{k=1}^N p_{k,t} f_{k,t}, y_t \right) \cdot \left( \sum_{j=1}^N p_{j,t} f_{j,t} - \sum_{j=1}^N q_j f_{j,t} \right)$$

$$= \max_q \sum_{t=1}^T \left( \sum_{j=1}^N p_{j,t} \tilde{\ell}_{j,t} - \sum_{j=1}^N q_j \tilde{\ell}_{j,t} \right)$$

$$= \sum_{t=1}^T \sum_{j=1}^N p_{j,t} \tilde{\ell}_{j,t} - \min_{i=1,\ldots,N} \sum_{t=1}^T \tilde{\ell}_{i,t}$$

where we denoted

$$\tilde{\ell}_{j,t} = \nabla \ell \left( \sum_{k=1}^N p_{k,t} f_{k,t}, y_t \right) \cdot f_{j,t}$$
Via the (signed) pseudo-losses

\[ \tilde{\ell}_{j,t} = \nabla \ell \left( \sum_{k=1}^{N} p_{k,t} f_{k,t}, y_t \right) \cdot f_{j,t} \]

it suffices to consider the following simplified framework.

At each round \( t = 1, 2, \ldots \),

- the experts provide forecasts \( f_{1,t}, \ldots, f_{N,t} \);
- the statistician picks convex weights \( p_t = (p_{1,t}, \ldots, p_{N,t}) \);
- the environment then determines, possibly with the knowledge of \( p_t \), a loss vector \( (\tilde{\ell}_{1,t}, \ldots, \tilde{\ell}_{N,t}) \)

The aim is to bound uniformly the regret

\[ R_T = \sum_{t=1}^{T} \sum_{j=1}^{N} p_{j,t} \tilde{\ell}_{j,t} - \min_{i=1,\ldots,N} \sum_{t=1}^{T} \tilde{\ell}_{i,t} \]
For all $j \in \{1, \ldots, N\}$, we pick $p_{j,1} = 1/N$ and for all $t \geq 2$,

$$p_{j,t} = \frac{\exp \left( -\eta \sum_{s=1}^{t-1} \tilde{\ell}_{j,s} \right)}{\sum_{k=1}^{N} \exp \left( -\eta \sum_{s=1}^{t-1} \tilde{\ell}_{k,s} \right)}$$

This strategy is known as performing exponentially weighted averages of the past cumulative losses of the experts (with fixed learning rate $\eta > 0$).

**Lemma.** Consider two real numbers $m \leq M$.

For all $\eta > 0$ and for all individual sequences $\tilde{\ell}_{j,t} \in [m, M]$,

$$R_T = \sum_{t=1}^{T} \sum_{j=1}^{N} p_{j,t} \tilde{\ell}_{j,t} - \min_{i=1,\ldots,N} \sum_{t=1}^{T} \tilde{\ell}_{i,t} \leq \frac{\ln N}{\eta} + \eta \frac{(M - m)^2}{8} T$$

References: Vovk '90; Littlestone and Warmuth '94
Proof of the regret bound

It relies on Hoeffding's lemma: for all random variables $X$ with range $[m, M]$, for all $s \in \mathbb{R}$,

$$\ln \mathbb{E}[e^{sX}] \leq s \mathbb{E}[X] + \frac{s^2}{8} (M - m)^2$$

For all $t = 1, 2, \ldots$,

$$-\eta \sum_{j=1}^{N} p_j t \tilde{\ell}_{j,t} = -\eta \sum_{j=1}^{N} \frac{\exp \left( -\eta \sum_{s=1}^{t-1} \tilde{\ell}_{j,s} \right)}{\sum_{k=1}^{N} \exp \left( -\eta \sum_{s=1}^{t-1} \tilde{\ell}_{k,s} \right)} \tilde{\ell}_{j,t}$$

$$\geq \ln \frac{\sum_{j=1}^{N} \exp \left( -\eta \sum_{s=1}^{t} \tilde{\ell}_{j,s} \right)}{\sum_{k=1}^{N} \exp \left( -\eta \sum_{s=1}^{t-1} \tilde{\ell}_{k,s} \right)} - \frac{\eta^2}{8} (M - m)^2$$

A telescoping sum appears and leads to

$$\sum_{t=1}^{T} \sum_{j=1}^{N} p_j t \tilde{\ell}_{j,t} \leq -\frac{1}{\eta} \ln \frac{\sum_{j=1}^{N} \exp \left( -\eta \sum_{s=1}^{T} \tilde{\ell}_{j,s} \right)}{N} + \eta \frac{(M - m)^2}{8} T.$$

$$\leq \min_{i=1, \ldots, N} \sum_{t=1}^{T} \tilde{\ell}_{i,t} + \frac{\ln N}{\eta}$$
We now discuss the obtained bound.

Recall that $[m, M]$ is the loss range.

The stated bound can be optimized in $\eta$:

$$R_T \leq \min_{\eta>0} \left\{ \frac{\ln N}{\eta} + \eta \frac{(M - m)^2}{8} T \right\} = (M - m) \sqrt{\frac{T}{2}} \ln N$$

for the (theoretical) optimal choice

$$\eta^* = \frac{1}{M - m} \sqrt{\frac{8 \ln N}{T}}$$

This choice depends on $M$ and $m$, which are sometimes not known beforehand, as well as on $T$, which may not be bounded (if the prediction game goes forever).

Since no fixed value of $\eta > 0$ ensures that $R_T = o(T)$, we still have no fully sequential strategy... but this can be taken care of.
The possible patches are, first, to resort to the “doubling trick.”

Alternatively, the learning rates of the exponentially weighted average strategy may vary over time, depending on the past: for \( t \geq 2 \),

\[
p_{j,t} = \frac{\exp\left(-\eta_t \sum_{s=1}^{t-1} \tilde{\ell}_{j,s}\right)}{\sum_{k=1}^{N} \exp\left(-\eta_t \sum_{s=1}^{t-1} \tilde{\ell}_{k,s}\right)}
\]

By a careful such adaptive choice of the \( \eta_t \), the following regret bound can be obtained:

\[
R_T \leq \Box (M - m)\sqrt{T \ln N} + \Box (M - m) \ln N
\]

where the \( \Box \) denote some universal constants.

We thus recover the same orders of magnitude for the regret bound.

References: Auer, Cesa-Bianchi and Gentile '02; Cesa-Bianchi, Mansour and Stoltz '07
However, these theoretically satisfactory solutions would not work well in practice.

This is what we do instead. (It is very different from techniques like cross-validation: we exploit the sequential fashion.)

The exponentially weighted average strategy $\mathcal{E}_\eta$ with fixed learning rate $\eta$ picks the convex combination $\mu_t(\eta)$, where

$$p_{j,t}(\eta) = \frac{\exp \left(-\eta \sum_{s=1}^{t-1} \hat{\ell}_{j,s}\right)}{\sum_{k=1}^{N} \exp \left(-\eta \sum_{s=1}^{t-1} \hat{\ell}_{k,s}\right)}$$

We denote its cumulative loss $\widehat{L}_t(\eta) = \sum_{s=1}^{t} \ell \left( \sum_{j=1}^{N} p_{j,s}(\eta)f_{j,s}, y_s \right)$

Based on the family of the $\mathcal{E}_\eta$, we build a data-driven meta-strategy which at each instance $t \geq 2$ resorts to

$$p_{t+1}(\eta_t) \quad \text{where} \quad \eta_t \in \arg \min_{\eta>0} \widehat{L}_t(\eta)$$

Reference: An idea of Vivien Mallet
Other natural variants: Focus on the most recent losses

Moving sums (with window of size $H$):

$$p_{j,t} = \exp \left( -\eta \sum_{s=\max\{1,t-H\}}^{t-1} \tilde{\ell}_{j,s} \right)$$

$$= \frac{\sum_{k=1}^{N} \exp \left( -\eta \sum_{s=\max\{1,t-H\}}^{t-1} \tilde{\ell}_{k,s} \right)}{\sum_{k=1}^{N} \exp \left( -\eta \sum_{s=1}^{t-1} (1 + \beta_{t-s}) \tilde{\ell}_{k,s} \right)}$$

One can prove that the regret is $\geq \Box T$ in the worst case.

Discounted losses (with discounts given by a sequence $\beta_t \downarrow 0$):

$$p_{j,t} = \exp \left( -\eta_t \sum_{s=1}^{t-1} (1 + \beta_{t-s}) \tilde{\ell}_{j,s} \right)$$

$$= \frac{\sum_{k=1}^{N} \exp \left( -\eta_t \sum_{s=1}^{t-1} (1 + \beta_{t-s}) \tilde{\ell}_{k,s} \right)}{\sum_{k=1}^{N} \exp \left( -\eta_t \sum_{s=1}^{t-1} (1 + \beta_{t-s}) \tilde{\ell}_{k,s} \right)}$$

Sublinear regret bounds hold for suitable sequences ($\beta_t$) and ($\eta_t$):

$$t \eta_t \to 0 \quad \text{and} \quad \eta_t \sum_{s \leq t} \beta_s \to 0$$

(We often take $\beta_s = \Box / s^2$ in the experimental studies.)
First study, continued

Forecasting of the air quality
### Oracles
(RMSE of the experts and of fixed combinations thereof)

<table>
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<tr>
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### Semi-sequential strategies
(RMSE of the strategies tuned with best parameters in hindsight)

<table>
<thead>
<tr>
<th>Original version</th>
<th>Moving sums ($H = 83$)</th>
<th>Discounts ($\beta_s = 1/s^2$)</th>
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<tbody>
<tr>
<td>21.47</td>
<td>21.37</td>
<td>21.31</td>
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### Fully sequential strategies
(RMSE of the original version of the strategy)

<table>
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<tr>
<td>21.47</td>
<td>21.77</td>
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</table>
Our strategies do not focus on a single expert. We knew it from the numerical performance.

But actually, the weights associated with the experts change quickly and significantly over time and do not converge (which illustrates in passing that the performance of the considered experts varies over time).

Convex weights output by the (original) strategy with best parameter $\eta$ in hindsight
<table>
<thead>
<tr>
<th>Framework</th>
<th>Convex weights</th>
<th>Linear weights</th>
<th>Summary / Second study</th>
<th>Conclusion</th>
</tr>
</thead>
</table>

A strategy to pick linear weights

It will ring a bell to the statisticians among you!
Linear combinations: Ridge regression (and the LASSO?)

The ridge regression was introduced in the 70s by Hoerl and Kennard; it was intensively studied since then in a stochastic setting.

We consider the case where $C \subseteq \mathbb{R}$ and $\ell(x, y) = (x - y)^2$.

The ridge regression resorts to linear combinations of the experts:

$$v_t \in \arg \min_{u \in \mathbb{R}^N} \left\{ \lambda \|u\|_2^2 + \sum_{s=1}^{t-1} \left( y_s - \sum_{j=1}^{N} u_j f_{j,s} \right)^2 \right\}$$

for some regularization parameter $\lambda > 0$.

It also exhibits a sublinear regret against individual sequences.

We do not know any such regret bounds for the LASSO yet.
Theorem. Consider a bound $B > 0$.

For all $\lambda > 0$, for all individual sequences of observations $y_t \in [-B, B]$ and of expert predictions $f_{j,t} \in [-B, B]$, for all $u \in \mathbb{R}^N$,

$$
\sum_{t=1}^{T} \ell \left( \sum_{j=1}^{N} v_{j,t} f_{j,t}, y_t \right) - \sum_{t=1}^{T} \ell \left( \sum_{j=1}^{N} u_{j} f_{j,t}, y_t \right) \\
\leq \lambda \|u\|_2^2 + 2NB^2 \left( 1 + \frac{N TB^2}{\lambda} \right) \ln \left( 1 + \frac{TB^2}{N\lambda} \right)
$$

$\lambda$ of the order of $1/\sqrt{T}$ is thus a good theoretical choice and leads to $O(\sqrt{T} \ln T)$ regret bounds.

Time-varying or data-driven parameters $\lambda_t$ can be considered (both for theoretical bounds or for the sake of practical performance).

References: Vovk '01; Azoury and Warmuth '01; Gerchinovitz '11
The interest of this method is that it can compensate for biases (in either direction) as the weights do not need to sum up to 1.

Even better, we can/should use it as a pre-treatment on each single expert and

- turn it into a modified expert with predictions $\gamma_t f_{j,t}$,
- performing on average almost as well as the best expert of the form $\gamma f_{j,t}$ for some constant $\gamma \in \mathbb{R}$.

This would improve greatly the predictions if there existed, for instance, an almost constant multiplicative bias of $1/\gamma$. 
First study, continued

Forecasting of the air quality
In our application to the prediction of air quality, we have extra sums over the stations of the network. E.g., the ridge regression is defined as

$$\mathbf{v}_t \in \arg \min_{\mathbf{u} \in \mathbb{R}^N} \left\{ \lambda \| \mathbf{u} \|_2^2 + \sum_{\tau=1}^{t-1} \sum_{s \in S_\tau} \left( y^s_{\tau} - \sum_{j=1}^{N} u_j f^s_{j,\tau} \right)^2 \right\}$$

One can show that $O(\sqrt{T} \ln T)$ regret bounds are still preserved.

The experts are indeed improved via the ridge pre-treatment. We illustrate this on the worst and best experts.

<table>
<thead>
<tr>
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<th>Pre-treated</th>
<th>Original</th>
<th>Pre-treated</th>
</tr>
</thead>
<tbody>
<tr>
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**Oracles**
(RMSE of the experts and of fixed combinations thereof)

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**Semi-sequential ridge regression**
(RMSE of the strategies tuned with best parameters in hindsight)

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**Fully sequential ridge regression**
(RMSE of the original version of the strategy)

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Our strategies do not focus on a single expert and the weights associated with the experts do not converge. [...]
Methodological summary
Methodological summary

1. **Build the $N$ experts** (possibly on a training data set) and pick another data set for the evaluation of our methods, with $T$ instances;

2. **Compute some benchmarks** and some reference **oracles**;

3. **Evaluate our strategies** when run with **fixed** parameters (i.e., with the best parameters in hindsight);

4. **The performance of interest is actually the one of the data-driven meta-strategies**.

We typically expect $T \geq 5N$ (or even $T \geq 10N$).

Hope arises when the oracles are 10% or 20% better than the methods used so far (e.g., the best expert when the latter is known in advance).

This usually requires the experts to be as different as possible.
Second study: Forecasting of exchange rates

Starting date: March 2012

Academic partner: Tomasz Michalski, HEC Paris

M.Sc. student involved over time:
  – Christophe Amat (5 months in 2013; from Ecole Polytechnique)

Associated publication: In preparation
The goal is to predict monthly averages $r_{t+1}$ of exchange rates based on few macro-economic indicators $x_{j,t}$ describing the state of the world:

- consumer price indexes (CPI);
- industrial production (Prod);
- monetary mass (Mon);
- required rates of return (“interest rates”, 3R).

They will give rise to four experts.

The prediction horizon is 1-month ahead.

A classical stochastic modeling is

$$\ln r_{t+1} = \ln r_t + \sigma (W_{t+1} - W_t)$$

for some Brownian motion $W$.

It is considered difficult to improve on it (Meese and Rogoff ’83).

It will give rise to the final expert (“random walk, RW”).
We denote by $r_t$ the averaged exchange rate of currency $A$ with respect to currency $B$.

We focus on the log–variation $y_{t+1} = \Delta_{t+1} = \ln r_{t+1} - \ln r_t$.

The stochastic modeling suggested the prediction $f_{0,t+1} = 0$.

The economic theory indicates that a given macro-economic indicator $j \in \{1, 2, 3, 4\}$ can be used to forecast the exchange rate according to

$$\hat{\Delta}_{j,t+1} = \ln x_{j,t}^B - \ln x_{j,t}^A \overset{\text{def}}{=} f_{j,t+1}$$

Using our methods we propose convex or linear combinations of the log–variations:

$$\hat{\Delta}_{t+1} = \sum_{j=0}^{4} u_{j,t+1} f_{j,t+1} = \sum_{j=1}^{4} u_{j,t+1} f_{j,t+1}$$
The predicted log–variations $\hat{\Delta}_t$ and exchange rates $\hat{\Delta}_t + \ln r_{t-1}$ are evaluated via their common RMSE:

$$\text{RMSE}_T = \sqrt{\frac{1}{T - t_0 + 1} \sum_{t=t_0}^{T} (\hat{\Delta}_t - \Delta_t)^2}$$

$$= \sqrt{\frac{1}{T - t_0 + 1} \sum_{t=t_0}^{T} ((\hat{\Delta}_t + \ln r_{t-1}) - \ln r_t)^2}$$

where $t_0 = 30$ allows a short training period.

We apply two (families of) strategies:

- **EG** (exponential weights on the gradients of the losses), as it leads to interpretable weights;
- the **ridge regression**, as it pushes in favor of the RW expert.
Some **orders of magnitude** for the prediction problems at hand are indicated below.

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<tr>
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<th>USD / GBP</th>
<th>JPY / USD</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time intervals</strong></td>
<td>Every month</td>
<td></td>
</tr>
<tr>
<td><strong>Period</strong></td>
<td>April 1973 – May 2013</td>
<td></td>
</tr>
<tr>
<td><strong>Time instances</strong> $T$</td>
<td>about 480</td>
<td></td>
</tr>
<tr>
<td><strong>Number of experts</strong> $N$</td>
<td>5 ($=1+4$)</td>
<td></td>
</tr>
<tr>
<td><strong>Median of the $\Delta_t$</strong></td>
<td>$1.48 \times 10^{-2}$</td>
<td>$1.57 \times 10^{-2}$</td>
</tr>
<tr>
<td>**Maximum of the $</td>
<td>\Delta_t</td>
<td>$**</td>
</tr>
</tbody>
</table>
## Results for USD / GBP

<table>
<thead>
<tr>
<th>Experts</th>
<th>RMSE</th>
<th>Oracle</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>RW</td>
<td>$2.47 \times 10^{-2}$</td>
<td>Best expert</td>
<td>$2.47 \times 10^{-2}$</td>
</tr>
<tr>
<td>CPI</td>
<td>$2.71 \times 10^{-2}$</td>
<td>Best $p$</td>
<td>$2.47 \times 10^{-2}$</td>
</tr>
<tr>
<td>3R</td>
<td>$2.84 \times 10^{-2}$</td>
<td>Best $u$</td>
<td>$2.46 \times 10^{-2}$</td>
</tr>
<tr>
<td>Prod</td>
<td>$2.59 \times 10^{-2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>$2.68 \times 10^{-2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**vs.**

| EG disc. | Semi-seq. | $2.39 \times 10^{-2}$ |
|          | Fully seq. | $2.42 \times 10^{-2}$ | ($-2.1\%$) |
| Ridge disc. | Semi-seq. | $2.35 \times 10^{-2}$ |
|            | Fully seq. | $2.36 \times 10^{-2}$ | ($-4.3\%$) |
# Results for JPY / USD

<table>
<thead>
<tr>
<th>Experts</th>
<th>RMSE</th>
<th>Oracle</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>RW</td>
<td>$2.82 \times 10^{-2}$</td>
<td>Best expert</td>
<td>$2.82 \times 10^{-2}$</td>
</tr>
<tr>
<td>CPI</td>
<td>$2.94 \times 10^{-2}$</td>
<td>Best $\mathbf{p}$</td>
<td>$2.82 \times 10^{-2}$</td>
</tr>
<tr>
<td>3R</td>
<td>$3.05 \times 10^{-2}$</td>
<td>Best $\mathbf{u}$</td>
<td>$2.79 \times 10^{-2}$</td>
</tr>
<tr>
<td>Prod</td>
<td>$2.88 \times 10^{-2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mon</td>
<td>$3.18 \times 10^{-2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

vs.

<table>
<thead>
<tr>
<th>EG disc.</th>
<th>Semi- seq.</th>
<th>$2.73 \times 10^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fully seq.</td>
<td>$2.79 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ridge disc.</th>
<th>Semi- seq.</th>
<th>$2.71 \times 10^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fully seq.</td>
<td>$2.73 \times 10^{-2}$</td>
</tr>
</tbody>
</table>
Other empirical studies

– Forecasting of the electricity consumption (EDF R&D)
– Forecasting of the production data of oil reservoirs (IFP–EN)

But time is over...

(Please consider inviting Pierre Gaillard for more details on our work with EDF R&D!)
References

In case you’re not bored to death (yet) by this topic!
The so-called “red bible!”

Prediction, Learning, and Games
Nicolò Cesa-Bianchi et Gábor Lugosi
I published a survey paper (containing this talk!) one year ago in the Journal de la Société Française de Statistique

Even better (or worse)—it is in French!